Defining Quality Measures for Validation and Generation of High-Order Tetrahedral Meshes

Abel Gargallo-Peiró[1], Xevi Roca[2], Jaime Peraire[2], and Josep Sarrate[1]

[1] Laboratori de Càlcul Numèric (LaCàN),
Universitat Politècnica de Catalunya, Barcelona 08034, Spain
{abel.gargallo,jose.sarrate}@upc.edu

[2] Department of Aeronautics and Astronautics,
Massachusetts Institute of Technology, Cambridge, MA 02139, USA
{xeviroca,peraire}@mit.edu

Summary. A procedure to quantify the distortion (quality) of a high-order mesh composed by curved tetrahedral elements is presented. The proposed technique has two main applications. First, it can be used to check the validity and quality of a high-order tetrahedral mesh. Second, it allows the generation of curved meshes composed by valid and high-quality high-order tetrahedral elements. To this end, we describe a method to simultaneously smooth and untangle high-order tetrahedral meshes by minimizing the proposed mesh distortion. Finally, we present several results to illustrate the two main applications of the proposed technique.

1 Introduction

In the last two decades, high-order methods have attracted a remarkable attention from the community of computational methods. This attention has been prompted by the potential of high-order methods to deliver higher accuracy with a lower computational cost than low order methods [1, 2, 3, 4, 5, 6, 7]. However, their adoption has been hampered by several technical issues such as the difficulty of developing robust implementations and the generation of 3D curved meshes. Note that a mesh for high-order methods is composed by straight-sided and potentially curved elements. The curved elements are required to approximate adequately the curved domain boundaries and therefore, to preserve the accuracy of high-order methods [8, 9, 10, 11, 12]. Hence, it is mandatory to develop techniques that allow the generation of valid 3D curved meshes to obtain full benefit of high-order methods.

A curved mesh is considered to be valid if each one of the elements can be transformed to a regular and straight-sided element (master) through

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a mapping that is differentiable, invertible and smooth (diffeomorphism). That is, the mapping between the master element and the curved element is expressed in terms of a differentiable function, and the curved element is non-folded (positive determinant of the mapping Jacobian). Moreover, a curved mesh is considered to be of high-quality if all the elements have a shape close to regular (smooth and well-conditioned Jacobian). Note that if an element is invalid, the determinant of the mapping Jacobian presents non-positive values. These non-positive values invalidate the change of variables used to compute the weak formulation integrals on a master element. Moreover, if an element has low-quality the Jacobian could be non-smooth or ill-conditioned. In that case, the approximation accuracy is degraded and the solution may be polluted by the introduced error [13]. Therefore, it is required to develop measures to quantify the validity and quality of a given 3D curved mesh.

Quality measures also have an alternative and significant application. They allow the use of optimization based techniques to repair non-valid meshes (untangle) and to improve the mesh quality (smooth) by minimizing (maximizing) the distortion (quality) of the mesh elements. Note that this technique allows the generation of high-order meshes with an a posteriori approach [14, 15, 16, 17, 18, 19]. That is, it allows the generation of meshes that might contain inverted or low-quality elements, and then untangle and smooth them a posteriori to ensure and enhance the mesh quality. Specifically, a high-order mesh can be obtained by generating first a linear mesh. Second, the linear mesh is converted to a high-order mesh by adding additional nodes and by curving the boundary elements. Finally, the converted mesh is untangled and smoothed to remove the non-valid (folded) and low-quality (distorted) elements. However, the application of this approach together with a mesh quality optimization has been hampered by the absence of 3D quality measures for high-order iso-parametric elements with degree superior than two. Note that the capability of generating valid curved 3D meshes is of the major importance for the community of high-order unstructured methods.

The main contribution of this work is to present a technique that allows extending any Jacobian based quality measure for linear 3D elements to high-order iso-parametric tetrahedra of any interpolation degree. In addition, we derive how to apply the proposed technique: to quantify the validity and quality of a given 3D curved mesh; and to generate curved meshes with an a posteriori approach based on the minimization of the mesh distortion. To illustrate these two main applications, we check the validity and quantify the quality of several curved meshes. Moreover, we present several curved meshes obtained with the presented a posteriori approach.

2 Related Work

One of the main issues in mesh generation is to quantify the validity of a mesh for computational purposes. For linear elements, a wide range of quality

measures have been developed [13, 20]. Specifically, this is an unresolved issue for high-order meshes. To address this issue, we develop a technique that allows extending to high-order elements the set of Jacobian-based measures for linear elements presented in [21, 22]. These measures allow determining the quality of a linear element in terms of the Jacobian of the iso-parametric mapping. Below we review the previous developed approaches to quantify the validity and quality of curved meshes.

On the one hand, different techniques have been proposed to determine the validity of a high-order mesh by means of checking the positivity of the Jacobian mapping from the master to the physical element. Specifically, it has been studied how to detect non-positive Jacobian determinants for B-spline based mappings [14, 15, 16, 17, 18] and quadratic iso-parametric elements [23, 24, 25]. Moreover, for higher interpolation degrees, in [26, 27] it is proposed to compute accurate bounds on Jacobian determinants of 2D and 3D curvilinear polynomial finite elements.

On the other hand, several approaches have been developed to quantify the quality of non-linear iso-parametric elements. For elements of quadratic degree, different definitions of distortion (quality) have been proposed for the planar [28, 29, 30, 31] and 3D [32, 33, 34] elements. The main difference of this work with the previous works is that we propose the definition of the distortion (quality) measure for tetrahedral elements (3D) of any interpolation degree. The proposed distortion is the $\mathcal{L}^2$-norm of the modification of a given point-wise Jacobian-based measure. This definition allows the detection of non-positive values of the Jacobian determinant of the master mapping for any interpolation degree. That is, if the quality is greater than zero, the master mapping is a local diffeomorphism in the integration points. It is important to point out that this work is an extension to 3D of the distortion measures for planar and surface elements presented in [35, 36].

The defined distortion measure allows both the validation of high-order tetrahedra and the generation of high-order meshes via an a posteriori optimization approach. The a posteriori approach has been previously used with success for B-spline mappings [14, 15, 16, 17, 18], and for iso-parametric elements [37, 19, 35, 38, 34, 36, 39]. Next, we highlight the two main types of approaches to generate high-order tetrahedral meshes by means of an a posteriori approach.

On the one hand, in [19, 38] two techniques to generate curved meshes by means of a solid mechanics analogy are presented (non-linear and linear elasticity, respectively). On the other hand, the curving of the mesh in the a posteriori approach can also be achieved by the optimization of a distortion (quality) measure. In [34] a technique is presented to generate meshes composed by tetrahedra of degree 2 via topology modification and optimization of two functions (scaled Jacobian measure and a ratio of the length and volume of the elements). In contrast, in [39] it is proposed to optimize a function that is able to avoid Jacobians that are too small or too high.

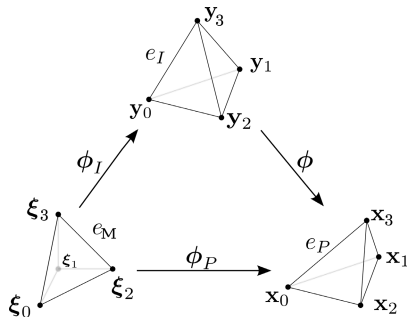

Fig. 1 Mappings between the master, ideal and physical linear elements

One of the main features of the optimization approach that we propose for high-order meshes is that it allows untangling (recovering from an inverted configuration) as well as smoothing meshes of any interpolation degree. Note that a technique to untangle high-order meshes is strictly necessary, since tangled elements appear in the a posteriori approach. The untangling capability is introduced explicitly in the formulation by means of the modification of the Jacobian-based measure presented in [40, 36]. This modification leads to an automatic and robust smoothing-untangling optimization approach. It is important to point out that the optimization process proposed here consists on a minimization of the defined high-order mesh distortion. Therefore, the resulting untangled mesh is valid, but it also maximizes the defined quality measure. Finally, we would like to highlight that the approach presented here to generate high-order tetrahedral meshes is an extension of the work developed for high-order 2D meshes [35], and high-order surface meshes [36].

3 Distortion and Quality Measures for High-Order Tetrahedral Elements

In this section, we first review the definition of Jacobian-based distortion measures for linear tetrahedral elements. In addition, we introduce the notation required for high-order elements. Finally, we propose a definition for a distortion and a quality measure for high-order tetrahedra in terms of the Jacobian-based distortion measures for linear elements.

3.1 Preliminaries

In this work, we propose to define the quality of a high-order tetrahedral element by means of a generalization of the Jacobian-based quality measures for linear elements presented in [21, 22]. To define a Jacobian-based measure for linear tetrahedra, three elements are required: the master, the ideal, and

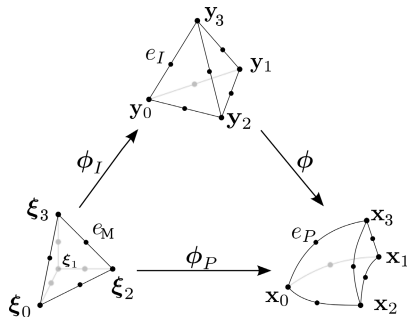

Fig. 2 Mappings between the master, ideal and physical high-order elements

the physical. The master element (e_R) is the element from which the isoparametric mapping is defined. The ideal tetrahedron (e_I) represents the target configuration. The physical (e_P) is the element to be measured. First, the mappings between the master and the ideal and the physical elements are obtained. By means of these applications, a mapping between the ideal and physical elements is determined by the composition (see Figure 1)

$$\boldsymbol{\phi} : e_I \xrightarrow{\boldsymbol{\phi}_I^{-1}} e_R \xrightarrow{\boldsymbol{\phi}_P} e_P.$$

The Jacobian of this affine mapping encodes the deviation of the physical element with respect to the ideal one. Hence, the distortion measure of the physical element is defined in terms of $\mathbf{D}\boldsymbol{\phi}$. These distortion measures, herein denoted by η, quantify the deviation of one or several features (shape, size, skewness, degeneracy,...) of the physical element with respect to the ideal element in a range scale $[1, \infty)$. The corresponding quality measure is defined as $q := 1/\eta \in [0, 1]$.

For the remaining of this work, we use the shape distortion measure [21]:

$$\eta(\mathbf{D}\boldsymbol{\phi}) = \frac{\|\mathbf{D}\boldsymbol{\phi}\|^2}{3|\sigma|^{2/3}}, \tag{1}$$

where $\|\cdot\|$ is the Frobenius norm, and $\sigma = \det(\mathbf{D}\boldsymbol{\phi})$. This distortion measure quantifies the deviation of the shape of the physical tetrahedron with respect to the ideal shape. To assign quality zero for degenerated elements (negative σ), σ in equation (1) is replaced by $\sigma_0 = (\sigma + |\sigma|)/2$. In this way, for negative values of σ, $\eta = \infty$ and therefore, $q = 0$.

To untangle meshes in the optimization procedure, we use the modification of the determinant σ proposed in [40]. This modification can be applied to Jacobian-based distortion measures where the determinant of the Jacobian appears in the denominator. Specifically, we replace σ in (1) by

$$\sigma_\delta(\sigma) = \frac{1}{2}\left(\sigma + \sqrt{\sigma^2 + 4\delta^2}\right), \tag{2}$$

where δ is a numerical parameter that has to be determined [40, 36]. Therefore, we consider a modified shape distortion measure,

$$\eta_\delta(\mathbf{D}\boldsymbol{\phi}) = \frac{\|\mathbf{D}\boldsymbol{\phi}\|^2}{3|\sigma_\delta|^{2/3}}. \tag{3}$$

It is important to point out that without the proposed modification, η has an asymptote when $\sigma = 0$ (where an element becomes non-valid). Note that it is required to modify σ to remove this asymptote and therefore, allow to the optimization procedure recovering from the non-valid configuration (tangled). For $\delta > 0$, $\sigma_\delta(\sigma)$ is a strictly increasing function, such that $\sigma_\delta(0) = \delta$ and that tends to 0 when σ tends to $-\infty$. Therefore, η_δ is a smooth function with no asymptotes. Moreover, for small values of δ, the minimum of η_δ is close to the valid minimum of η_0.

3.2 Notation for Iso-Parametric High-Order Tetrahedra

Let e_P be a nodal high-order tetrahedron of interpolation degree p determined by $n_p = (p+1)\cdot(p+2)\cdot(p+3)/6$ nodes with coordinates $\mathbf{x}_i \in \mathbb{R}^3$, for $i = 1, \ldots, n_p$. We select a node distribution that provides a quasi-optimal Lebesgue constant [41]. Given a master element e_R with nodes $\boldsymbol{\xi}_i \in \mathbb{R}^3$, being $i = 1, \ldots, n_p$, we consider the basis $\{N_i\}_{i=1,\ldots,n_p}$ of nodal shape functions (Lagrange interpolation) of degree p. Then, the high-order isoparametric mapping from e_R to e_P can be expressed as:

$$\begin{aligned} \boldsymbol{\phi}_P : e_R \subset \mathbb{R}^3 &\longrightarrow e_P \subset \mathbb{R}^3 \\ \boldsymbol{\xi} \quad &\longmapsto \mathbf{x} = \boldsymbol{\phi}_P(\boldsymbol{\xi}) = \sum_{i=1}^{n_p} \mathbf{x}_i N_i(\boldsymbol{\xi}). \end{aligned} \tag{4}$$

The Jacobian of the isoparametric mapping (4) is the matrix

$$\mathbf{D}\boldsymbol{\phi}_P(\boldsymbol{\xi}) = \sum_{i=1}^{n_p} \mathbf{x}_i \boldsymbol{\nabla} N_i(\boldsymbol{\xi}) \in \mathbb{R}^3 \times \mathbb{R}^3. \tag{5}$$

3.3 Definition of Distortion and Quality

In this section, we define the distortion and quality measure for a high-order element in terms of a Jacobian-based distortion measure for linear elements. To this end, analogously to the linear case, we consider three elements: the master e_R, the ideal e_I, and the physical e_P. The master element allows the characterization of the ideal and physical elements in terms of a master mapping. Specifically, it is required to determine a distribution of points on the master element and the corresponding coordinates of the nodes in

the ideal and physical element. On the one hand, the coordinates of the nodes of the ideal element (see Section 4.1 for more details) are chosen to determine a straight-sided high-order tetrahedron. Therefore, $\boldsymbol{\phi}_I$ is an affine mapping. On the other hand, the coordinates of the nodes of the physical element determine, in general terms, a curved element. Hence, the mapping $\boldsymbol{\phi}_P$ presented in Equation (4), is not affine in general, see Figure 2. Moreover, $\boldsymbol{\phi} = \boldsymbol{\phi}_P \circ \boldsymbol{\phi}_I^{-1}$ is also not affine, and the Jacobian matrix is not constant. Specifically, for a point $\mathbf{y}$ on the ideal element, the expression of the Jacobian is:

$$\mathbf{D}\boldsymbol{\phi}(\mathbf{y}) = \mathbf{D}\boldsymbol{\phi}_P(\boldsymbol{\phi}_I^{-1}(\mathbf{y})) \cdot \mathbf{D}\boldsymbol{\phi}_I^{-1}(\mathbf{y}). \tag{6}$$

Similar to the linear case, we want to define a distortion measure based on the Jacobian matrix of $\boldsymbol{\phi}$. However, the Jacobian of a high-order element is not constant. Nevertheless, the Jacobian on a point allows measuring the local deviation between the ideal and the physical element. Thus, we can obtain an elemental distortion measure by considering the $\mathcal{L}^2$-norm of the Jacobian-based distortion measure on the ideal element.

We define the ideal mesh $\mathcal{M}_I$ of a mesh $\mathcal{M}$ as the set of ideal elements that correspond to each physical element. Next, we define the scalar product of two scalar functions on $\mathcal{M}_I$ as

$$\langle f, g\rangle_{\mathcal{M}_I} := \sum_{i=1}^{n_E} \langle f, g\rangle_{e_{I_i}}, \tag{7}$$

expressed in terms of their inner product on the ideal element,

$$\langle f, g\rangle_{e_I} := \int_{e_I} f(\mathbf{y})\ g(\mathbf{y}) \mathrm{d}\mathbf{y}, \tag{8}$$

where e_{I_i} is the ideal of element e_{P_i}, and n_E is the number of elements of the mesh. The norms corresponding to these scalar products are

$$\|f\|_{\mathcal{M}_I} := \sqrt{\langle f, f\rangle_{\mathcal{M}_I}}, \tag{9}$$

$$\|f\|_{e_I} := \sqrt{\langle f, f\rangle_{e_I}}, \tag{10}$$

Remark 1. We choose e_I as a valid straight-sided tetrahedron. That is, $\boldsymbol{\phi}_I$ is an affine mapping and therefore, a global diffeomorphism. In this way, we can use the change of variable determined by $\boldsymbol{\phi}_I$ to compute the scalar product as:

$$\langle f, g\rangle_e := \int_{e_R} f(\boldsymbol{\phi}_I(\boldsymbol{\xi}))\ g(\boldsymbol{\phi}_I(\boldsymbol{\xi}))\ |\det \mathbf{D}\boldsymbol{\phi}_I(\boldsymbol{\xi})|\ \mathrm{d}\boldsymbol{\xi}. \tag{11}$$

To compute this integral, we use a numerical quadrature that ensures that polynomials of degree $(d+3)p - d$ and d variables are integrated exactly, where $d = 3$ is the number of spatial dimensions. Specifically, the quadrature use $((d+3)p - d - 1)/2$ integration points as specified in [42].

Next, we define the distortion at a point $\mathbf{y} \in e_I$ for a high-order element with nodes $\mathbf{x}_1, \ldots, \mathbf{x}_{n_p} \in \mathbb{R}^3$, as:

$$\eta_{\boldsymbol{\phi}} := \eta_\delta \left(\mathbf{D}\boldsymbol{\phi} \right). \tag{12}$$

We point out that $\eta_{\boldsymbol{\phi}}$ is a function of $\mathbf{y}$, and that it also depends on $\mathbf{x}_1, \ldots, \mathbf{x}_{n_p}$, since $\boldsymbol{\phi}$ does. Now, we can define the corresponding distortion and quality measures for a high-order element:

Definition 1. The *distortion measure for a high-order 3D element* is

$$\eta_{e_P} := \frac{\|\eta_{\boldsymbol{\phi}}\|_{e_I}}{\|1\|_{e_I}}, \tag{13}$$

where η_{e_P} is a function of the element nodes $\mathbf{x}_1, \ldots, \mathbf{x}_{n_p}$, since $\eta_{\boldsymbol{\phi}}$ is. Note that $\|1\|_{e_I}$ is the area of the ideal element.

Definition 2. The *quality measure for a high-order planar element* is $q_{e_P} := 1/\eta_{e_P}$.

4 Applications

In this section, we analyse the two main applications of the defined distortion and quality measures. First, we overview the main features that allow using these measures to validate a given high-order tetrahedral mesh. Second, we set an optimization framework in order to generate high-order tetrahedral meshes in an a posteriori procedure.

4.1 Validation of High-Order Tetrahedral Meshes

One of the main applications of distortion (quality) measures is to check that the mesh is valid to perform a numerical simulation. Specifically, a quality measure has to properly detect if an element it is non-valid (and assign 0 value). Moreover, the measure has to penalize the deviation of the element with respect to the target ideal (and assign value 1 to the ideal).

To measure the validity of elements, δ is set to 0 in (12). Therefore, if there is a point $\mathbf{y}$ where the Jacobian is non-positive ($\sigma \leq 0$), then $\eta_{\boldsymbol{\phi}}(\mathbf{y}) = \infty$. Hence, $\eta_{e_P} = \infty$. Conversely, if the physical element is the ideal, $\boldsymbol{\phi}$ is the identity. Then, the point-wise distortion $\eta_{\boldsymbol{\phi}}(\mathbf{y})$ is 1 for all $\mathbf{y} \in e_I$. Thus, by Definition 1, the element distortion η_{e_P} is also 1. Summarizing, we state the following remark:

Remark 2. The distortion measure η_{e_P} for high-order tetrahedral elements has image $[1, \infty)$, where 1 corresponds to the ideal configuration and ∞ to a non-valid one. Hence, by Definition 2, q_{e_P} has image $[0, 1]$, where 0 corresponds to an inverted element, and 1 to the ideal one.

The measures defined in Section 3.3 can accommodate different ideal elements. We select two different type of ideals depending on the framework of the computation. On the one hand, for absolute quality measurement, we select the equilateral tetrahedron as ideal (standard ideal for isotropic meshes [21]). The equilateral tetrahedron is used to obtain an absolute value of the quality, where all the elements are measured with respect the same target tetrahedron.

On the other hand, for optimization purposes we use a relative quality measurement. That is, we assume that we have an initial linear mesh, and for each high-order curved element we select the corresponding initial straight-sided element as ideal. It is important to point out that we assume that the initial linear mesh verifies the geometrical constraints for the numerical simulation, see Section 1. Therefore, in the a posteriori mesh generation procedure, our final goal is to optimize the location of the inner nodes to obtain a valid (without tangled elements) high-order mesh composed by elements with a shape similar to the one in the initial linear mesh.

4.2 Generation of Curved and High-Order Tetrahedral Meshes

The second application for the defined distortion measures is the generation of high-order meshes via an a posteriori approach [19]. The a posteriori mesh generation approach is composed by several steps. First, we generate a linear tetrahedral mesh ($\mathcal{M}_I$). The elements of this mesh have the shape and the size that hold the requirements for the desired numerical computation. Second, we increase the degree of the elements and we curve the boundary faces to fit the CAD geometry. For each high-order element, we set as ideal the corresponding straight-sided element in the linear mesh $\mathcal{M}_I$. Note that the boundary elements can be inverted or present low quality. Finally, we smooth and untangle the high-order curved mesh by minimizing the distortion of the curved mesh with respect to the straight-sided ideal.

It is important to point out that the surface mesh itself can be tangled when the faces are curved to fit the geometry. If there is a non-valid surface element, the 3D tetrahedron can not recover from the inversion. Therefore, the a posteriori approach that we use for 3D high-order meshes is composed by two optimization steps. First, we smooth and untangle the surface mesh [36]. Second, once the surface mesh is valid, we fix the nodes of the boundary surfaces and then we optimize the volume mesh. In this section, we present an algorithm to optimize the distortion of high-order tetrahedral meshes in order to obtain valid final meshes.

The main goal of a simultaneous smoothing and untangling method is to obtain high-quality meshes composed by valid (non-inverted) elements. Note that the best possible result, can be characterized in terms of the distortion

measure. That is, given a distortion measure η and a mesh $\mathcal{M}$ composed by n_N nodes and n_E elements, the node location is ideal if

$$\eta_{\boldsymbol{\phi}} = 1, \qquad \forall \mathbf{y} \in \mathcal{M}_I, \quad \text{(ideal mesh distortion)} \tag{14}$$

where $\eta_{\boldsymbol{\phi}}$ is the point-wise distortion presented in (12), and $\mathcal{M}_I$ is the mesh composed by the ideal elements (initial straight-sided mesh). That is,

$$\eta_{\boldsymbol{\phi}}(\mathbf{y}; \mathbf{x}_{j,\hat{1}}, \ldots, \mathbf{x}_{j,\hat{n}_p}) = 1, \qquad \forall \mathbf{y} \in e_{I_j}, \; j = 1, \ldots, n_E, \tag{15}$$

where e_{I_j} is the ideal element corresponding to the element e_j, and the pairs $(j, \hat{i})$ in $\mathbf{x}_{j,\hat{i}}$ identify the local $\hat{i}$-th node of element j with their global mesh number i. However, for a fixed mesh topology and a given surface, the node location that leads to an ideal mesh distortion is in general not achievable. That is, the constraints in Equation (14) cannot be imposed strongly and therefore, we enforce the ideal mesh distortion in the least-squares sense.

For a given mesh topology and a set of fixed nodes (nodes on the surface boundary), we formulate the non-linear least-squares problem in terms of the coordinates of a set of free nodes (nodes in the interior of the domain). To this end, we reorder the coordinates of the nodes, $\mathbf{x}_i$, in such a way that $i = 1, \ldots, n_F$ are the indices corresponding to the free nodes, and $i = n_F + 1, \ldots, n_N$ correspond to the fixed nodes. Defining

$$f(\mathbf{x}_1, \ldots, \mathbf{x}_{n_F}; \mathbf{x}_{n_F+1}, \ldots, \mathbf{x}_{n_N}) := \frac{1}{2} \|\eta_{\boldsymbol{\phi}} - 1\|^2_{\mathcal{M}_I},$$

we can formulate the mesh optimization problem as

$$\min_{\mathbf{x}_1, \ldots, \mathbf{x}_{n_F}} f(\mathbf{x}_1, \ldots, \mathbf{x}_{n_F}; \mathbf{x}_{n_F+1}, \ldots, \mathbf{x}_{n_N}). \tag{16}$$

In this work, we solve this least-squares problem by means of a non-linear iterative Gauss-Seidel method, as proposed in [36].

5 Results

In this section, we present two examples in order to illustrate the properties of the proposed quality measures. First, we show that the defined measures allow checking the validity of high-order tetrahedral meshes. Second, we show the a posteriori mesh generation approach. To this end, we present the main steps for the generation of a a mesh of interpolation degree 4 of the exterior domain of a Falcon aircraft. The proposed algorithm has been implemented in C++ in the meshing environment EZ4U [43, 44].

5.1 Validation of High-Order Tetrahedral Meshes

In this section, we illustrate that the defined measure is capable of quantifying the validity of a high-order tetrahedron. Specifically, we show that the measure detects when an element is valid or not. Moreover, it properly determines the deviation of a given element with respect to the considered ideal. Finally, we show that the presented untangling procedure is robust and can untangle high-order meshes composed by inverted elements.

We consider the geometry presented in Figure 3(a), and we generate a coarse tetrahedral mesh. Then, we generate three meshes of interpolation degree 2, 4 and 6 with the same topology, and we tangle them, see Figures 3(a), 3(b) and 3(c). In Figures 3(d), 3(e) and 3(f) we present the smoothed meshes. Note that the optimization approach generates valid and high-quality meshes. In Table 1 we present the quality statistics of the obtained mesh, where we see that all the inverted elements have been untangled and the overall quality statistics improved. The quality statistics and the displayed quality in Figure 3 is computed taking as ideal the initial linear mesh. Therefore, we observe

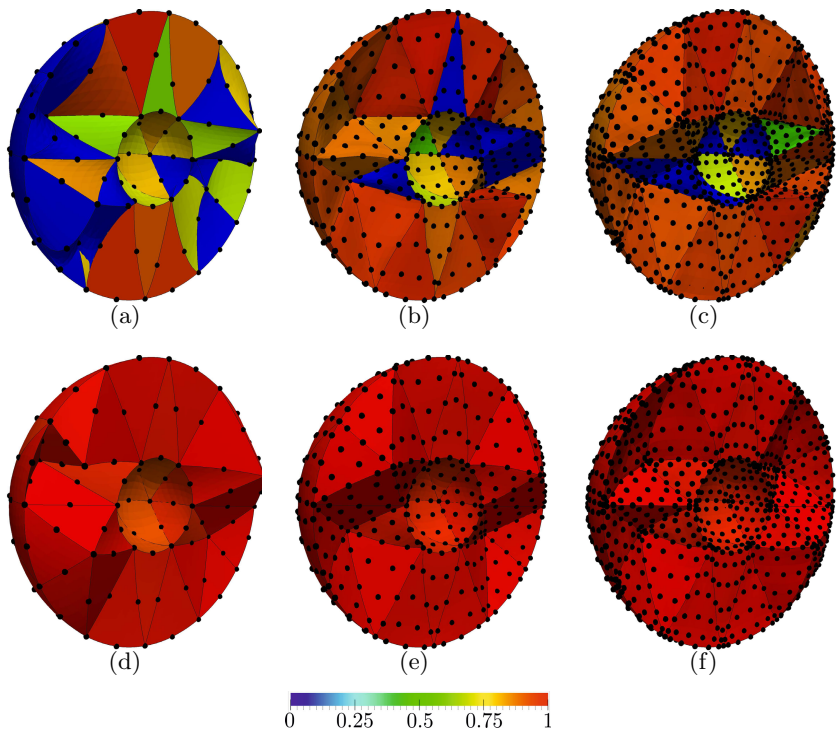

Fig. 3 Tetrahedral meshes of degree 2, 4 and 6 colored according to the shape quality measure (initial straight-sided ideal) on a hollow sphere: (a-c) initial meshes, and (d-f) smoothed meshes

Table 1 Shape quality statistics for a hollow sphere. The mesh is composed by 160 elements.

Degree	#nodes	Mesh	Figure	Min. Q.	Max. Q.	Mean Q.	Std. Dev.	Tang. el.
2	302	Initial	3(a)	0.00	0.97	0.51	0.40	59
		Smoothed	3(d)	0.91	0.99	0.98	0.02	0
4	2042	Initial	3(b)	0.00	0.97	0.81	0.28	16
		Smoothed	3(e)	0.95	1.00	0.99	0.01	0
6	6502	Initial	3(c)	0.00	0.96	0.81	0.27	15
		Smoothed	3(f)	0.95	1.00	0.99	0.01	0

that in Figures 3(d), 3(e) and 3(f) almost all the elements are of quality one. That is, the optimized mesh is close to the straight-sided one.

5.2 Generation of Curved and High-Order Tetrahedral Meshes

The main goal of this example is to show the a posteriori mesh generation process on a CAD geometry. We choose a Falcon aircraft to generate a mesh of degree 4 with valid and high-quality elements on the exterior domain. As it has been previously introduced, first we generate a linear mesh on the geometry, presented in Figure 4(a). We choose this mesh as the ideal, since we want to preserve the features (shape, size, isotropy...) that have been assigned to the linear mesh. It is important to point out that we do not explicitly impose that the element size is preserved. We use the shape quality measure presented in Equation 3. That is, we only impose that the shape of the initial linear mesh is preserved. In practice, this approach leads to meshes that resemble the element size of the initial linear mesh. To impose explicitly that the element size is preserved, it is required to use a distortion that combines at the same time a shape and size measure as we proposed in [36].

Once a linear mesh is generated, the degree of the mesh is increased and the boundary faces are curved to fit the CAD representation. In Figure 4(b) we present the initial curved mesh. This process leads to several non-valid and low-quality elements, see Table 2. In Figure 5(a), we illustrate some inverted elements that are originated after curving the boundary faces.

Table 2 Shape quality statistics of a mesh of interpolation degree 4 for a Falcon aircraft. The mesh is composed by 27511 elements and 317854 nodes.

Mesh	Quality	Figure	Min. Q.	Max. Q.	Mean Q.	Std. Dev.	Tang. el.
Initial	Absolute	4(b),6(a)	0.00	1.00	0.77	0.11	9
	Relative	6(c)	0.00	1.00	0.99	0.02	9
Smoothed	Absolute	4(d),6(b)	0.21	1.00	0.78	0.11	0
	Relative	6(d)	0.32	1.00	0.99	0.04	0

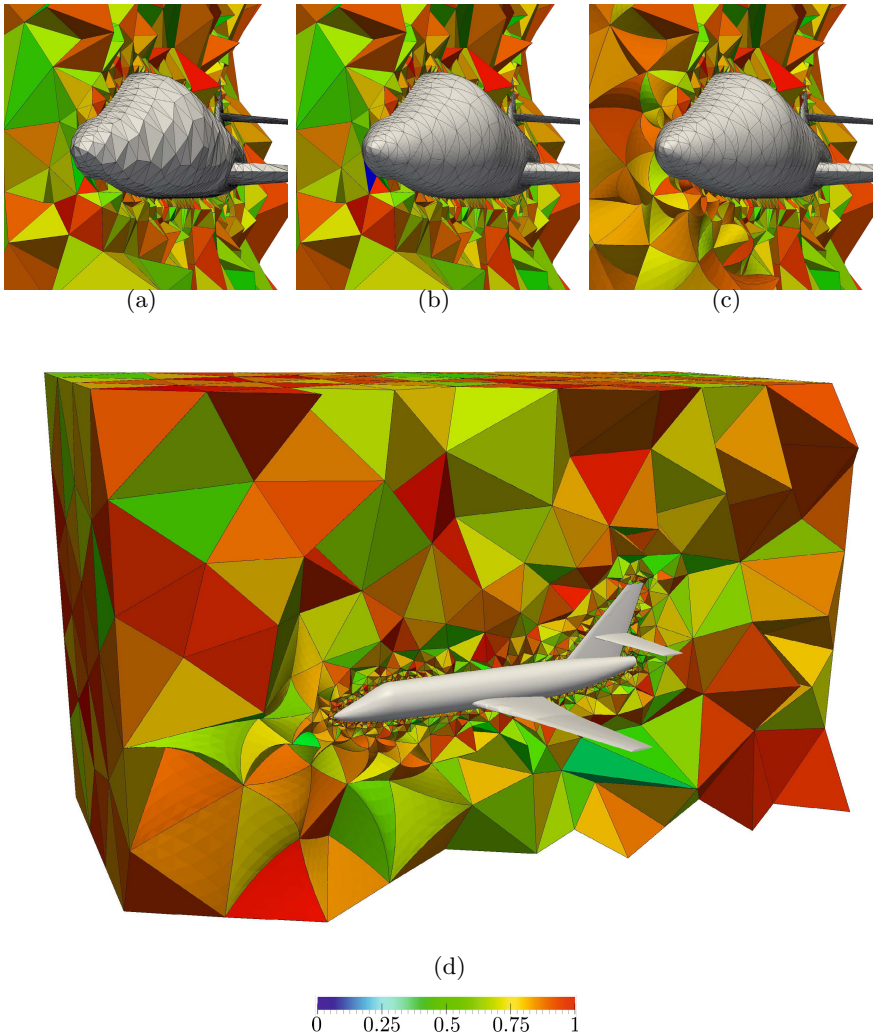

Fig. 4 Tetrahedral meshes of interpolation degree 4 colored according to the shape quality (equilateral ideal) measure on a Falcon aircraft: (a) initial straight-sided mesh, (b) initial curved mesh, and (c) smoothed mesh.

Next, we optimize the mesh. To this end, we relocate the nodes on the CAD surface [36], and afterwards we optimize the 3D mesh by means of the approach presented in Section 4. In Figures 4(c) and 4(d) we present the optimized mesh. Note that all the non-valid elements of the mesh have been repaired. Specifically, in Figure 5(b) we show the smoothed set of elements that correspond to the originally inverted ones in Figure 5(a).

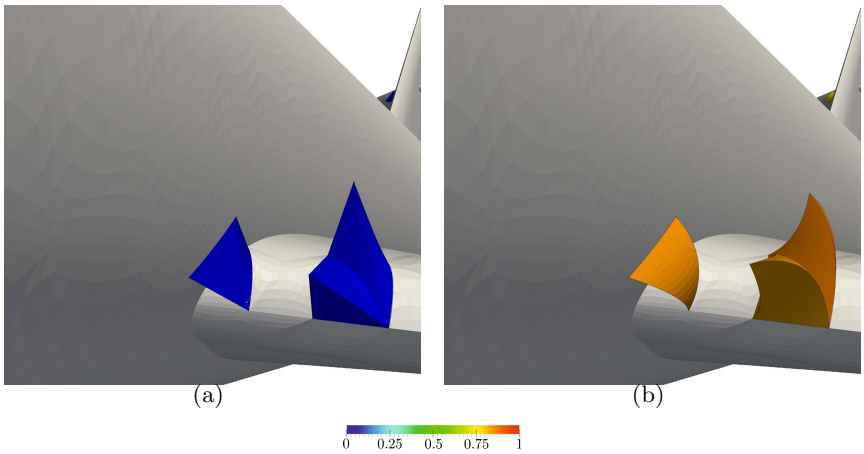

Fig. 5 Tetrahedral meshes of degree 4 colored according to the shape quality measure (equilateral ideal) on a Falcon aircraft. Detail of the mesh where inverted elements can be observed: (a) initial curved mesh, and (b) smoothed mesh.

It is important to point out that we illustrate the meshes displaying the quality with respect to two different ideals: the equilateral tetrahedron, and the initial straight-sided tetrahedra. The optimization is developed taking as ideal the initial mesh, since we assume that this mesh has the necessary computational requirements. However, the equilateral tetrahedron is used in isotropic meshes to allow the comparison of the qualities between different elements.

On the one hand, in Figures 6(b) and 6(b) the elements are colored taking as ideal the equilateral tetrahedron. On the other hand, Figures 6(c) and 6(d) are colored with the quality respect to the initial straight-sided mesh. In Figure 6(c), we observe that almost all the elements have initially quality one. This is due to the fact that except the elements on the boundary, the rest are still straight-sided. On the contrary, the curved elements have lower quality and they can even be non-valid. In Figure 6(d), the elements have been smoothed in order to untangle the inverted ones.

Table 2 presents the quality statistics of the generated meshes. It is important to point out that the final mesh is composed by valid and high-quality elements. Moreover, the minimum quality respect to the straight-sided mesh has been improved from 0 to 0.32. That is, the proposed procedure has generated a high-order curved mesh with high-quality elements that preserves the geometrical features prescribed in the linear mesh.

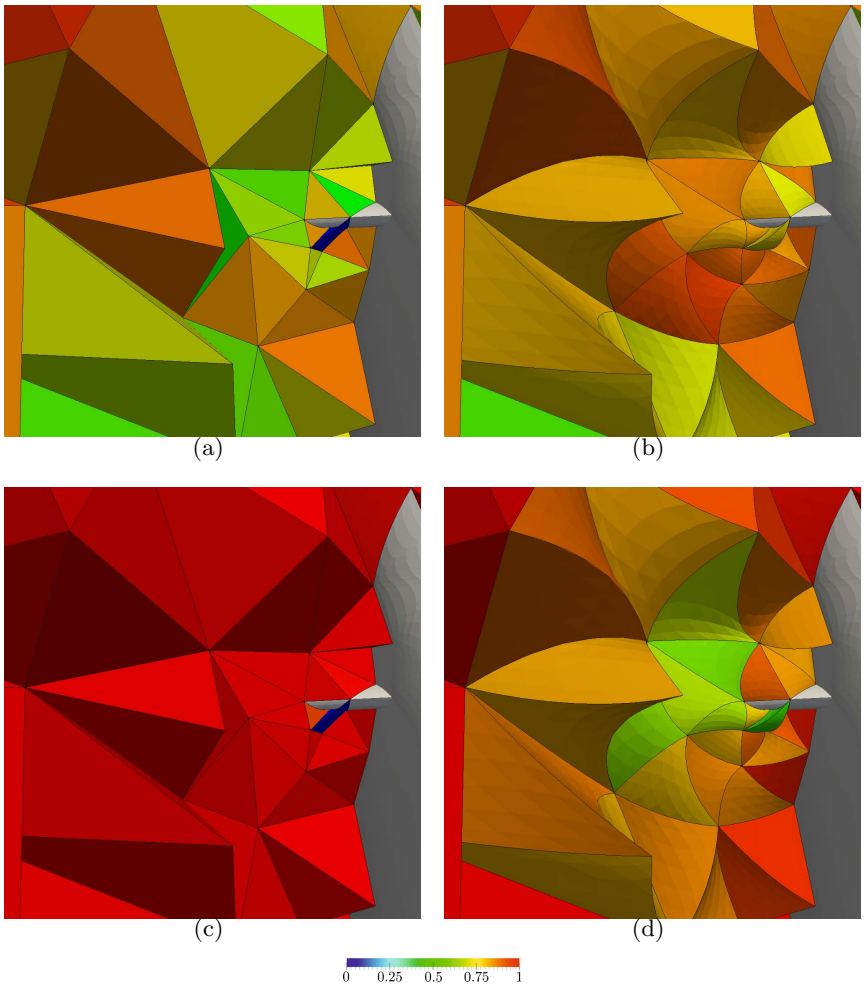

Fig. 6 Detail of the tetrahedral meshes of interpolation degree 4 on a Falcon aircraft. (a,c) Initial meshes colored with respect to the equilateral and initial straight-sided ideals. (b,d) Smoothed meshes colored with respect to the equilateral and initial straight-sided ideals.

6 Concluding Remarks

In this work, we present a new technique to define distortion (quality) measures for high-order tetrahedral meshes. The proposed definition is valid for any interpolation degree and allows detecting the validity of a high-order tetrahedron. Moreover, the defined distortion measure can be used to generate 3D curved high-order meshes. Specifically, the distortion is capable of smoothing and untangling high-order meshes by means of an optimization

procedure. Note that this is the final step of a curved meshing a posteriori approach. Finally, we show that this technique allows the generation of curved meshes from real CAD geometries.

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