Analysis of Hessian Recovery Methods for Generating Adaptive Meshes

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Summary. We study adaptive meshes which are quasi-uniform in a metric generated by the Hessian of a P_1 finite element function. We consider three most efficient methods for recovering this Hessian, one variational method and two projection methods. We compare these methods for problems with anisotropic singularities to show that all Hessian recovery methods result in acceptable adaptive meshes although the variational method gives a smaller error.

1 Introduction

Generation of adaptive meshes is now the standard option in most software packages. Contrary to uniform meshes, adaptive meshes with the same number of elements result in much more accurate solutions of PDEs. In this article, we consider unstructured adaptive meshes which are frequently anisotropic.

One way to generate the adaptive mesh is to define a metric that reflects problem specifics. *A posteriori* error estimates or some features of a problem solution can be used to define this metric. Different error estimates for anisotropic meshes can be found in [8, 9] (see also references therein). In this article, we focus on the maximum norm of the interpolation error for P_1 finite element solutions. In this case, the metric is induced by the Hessian (matrix of second derivatives) of the exact solution. Since the exact solution is not available, the Hessian must be recovered from the finite element solution.

Among many methods for recovering the Hessian, we selected three most efficient methods, one variational method [4] and two projection methods [3, 12]. We compare these methods for problems with anisotropic singularities. This research topic was inspired by the statement made in article [7] that the L_2-projection method was preferable for the Delaunay-type mesh generation algorithm considered there. In this article, we consider different mesh generation algorithm and show that all Hessian recovery methods result in acceptable adaptive meshes although the variational method gives a smaller error.

The Hessian recovery methods are illustrated with 2D examples. Their extension to 3D tetrahedral meshes is straightforward. These methods have linear complexity with respect to the number of mesh nodes.

The paper outline is as follows. In Section 2, we describe our metric-based mesh generation method. In Section 3, we consider the metric induced by the Hessian of a continuous piecewise linear mesh function and discuss different methods to recover this discrete Hessian. In Section 4, we analyze numerically these methods for problems with anisotropic singularities.

2 Metric-based Mesh Generation

Let Ω be a polygonal domain, and $M(x)$ be a symmetric positive definite 2×2 matrix for any $x = (x_1, x_2)$ in Ω. In metric given by M, the volume of a domain $D \subset \Omega$ and the length of a curve ℓ are defined as follows:

$$|D|_M = \int_D \sqrt{\det M(x)}\,\mathrm{d}x, \qquad |\ell|_M = \int_0^1 \sqrt{(M(\gamma(t))\gamma(t), \gamma(t))}\,\mathrm{d}t,$$

where $\gamma(t) : [0, 1] \to \Re^2$ is a parametrization of ℓ.

Let Ω_h be a triangular mesh covering the domain Ω. In this article, we analyze meshes that are quasi-uniform in metric $M(x)$ (or M-quasi-uniform). These meshes have a number of useful approximation properties when the metric is connected with *a posteriori* error estimates or solution features. A short summary of these properties is given in the next section.

There are many methods for generating a M-quasi-uniform mesh (see, for example, [7]). In this article, we focus on iterative methods that use a sequence of local mesh modifications to generate a global mesh. The list of mesh modifications includes alternation of topology with node deletion/insertion end edge swapping, and node movement. To describe the method, we need the notion of a *mesh quality*. For a triangle Δ of Ω_h, we denote by $p(\Delta)$ the total length of its edges (perimeter) in the metric M. Then, the mesh quality is defined by

$$Q(\Omega_h) = \min_{\Delta \in \Omega_h} Q(\Delta), \tag{1}$$

where $0 < Q(\Delta) \leqslant 1$ is the quality of triangle Δ (see [4] for more details),

$$Q(\Delta) = 12\sqrt{3}\frac{|\Delta|_M}{p(\Delta)^2} F\left(\frac{p(\Delta)}{3h_*}\right). \tag{2}$$

Here $F(t)\colon \Re^+ \to [0, 1]$ is a continuous smooth function, $0 \leqslant F(t) \leqslant 1$, with the only maximum at point 1, $F(1) = 1$, and such that $F(0) = F(+\infty) = 0$. Parameter h_* is equal to the size of elements in a mesh consisting of N_* equilateral (in the metric M) triangles. Thus, h_* is a simple function of $|\Omega|_M$ and N_*. The last factor in (2) controls the size of the element, whereas the remaining factors control its shape.

Since the mesh quality is equal to the quality of the worst triangle, the local mesh modifications are applied to this triangle. We accept the first operation that minimizes the mesh quality. If none of the mesh modifications increases mesh quality, the triangle is temporary placed in a special list and the next worst triangle is processed. When the number of elements in the list exceeds some threshold, all of them are released back into the mesh.

This iterative method requires an initial mesh. The initial mesh may be arbitrary and very coarse. The parameter h_* provides simple control over the number of mesh elements. The bigger mesh quality $Q(\Omega_h)$, the closer the number of mesh elements to N_* which is the given number. In a computer program, we terminate the iterative method when $Q(\Omega_h)$ becomes bigger than 0.7.

3 Recovery of Metric from Hessian of Discrete Solution

The special choice of the metric $M(x)$ will allow us to generate meshes with desired properties. The most desirable property for applications is minimization of certain norm of the solution error. In this article, we consider the interpolation error, $u - \mathcal{I}_h u$, where u is a known function and $\mathcal{I}_h$ is the piecewise linear interpolation operator.

A priori error bounds for the maximal norm on optimal meshes have been studied since the beginning of 90s [6]. Recall that the optimal mesh is defined as the mesh that minimizes the interpolation error among conformal meshes with a bounded number of elements. The main difficulty in analysis is possible anisotropy of the optimal mesh.

The lower upper bound on the error for general discrete spaces was obtained by V.Tikhomirov in 1960. In [1, 10], we analyzed a more narrow class of spaces of continuous piecewise linear functions on conformal triangulations and showed that the optimal meshes still provide the same error decay which is reciprocal to the number of elements N_*. In [11], our analysis has been extended to L_p-norm, where $p > 0$. For sufficiently smooth solutions ($u \in C^2(\bar{\Omega})$) and $0 < p \leqslant +\infty$, the asymptotical error is

$$\|u - \mathcal{I}_h u\|_{L_p(\Omega)} \sim N_*^{-1}.$$

Moreover, in [1, 10] we proved that meshes which are only quasi-uniform in the metric derived from the Hessian of u still result in the optimal interpolation error. Now, we describe this metric in more details.

Let $H(x)$ be the Hessian of u,

$$H = \{H_{ij}\}_{i,j=1}^2, \qquad H_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad i, j = 1, 2.$$

We consider its spectral decomposition $H(x) = W^T(x)\Lambda(x)W(x)$ and generate the metric $M(x)$ as follows:

$$M(x) = W^T(x)|\Lambda(x)|W(x), \qquad |\Lambda(x)| = \mathrm{diag}\{|\lambda_1(x)|,\ |\lambda_2(x)|\}, \tag{3}$$

where $\lambda_i(x)$ are eigenvalues of $H(x)$. To emphasize distinctive nature of this metric, we shall write $|H(x)|$ instead of $M(x)$.

In a practical adaptive computation, the unknown solution Hessian $H(x)$ must be replaced by its discrete approximation H^h. In [1, 10], we have formulated sufficient conditions for approximation H^h under which the mesh quasi-uniform in the metric $|H^h|$ is also quasi-uniform in the metric $|H(x)|$. If the initial mesh, used to recover H^h, is far from the optimal one, the discrete Hessian may not approximate the continuous one. In this case, the problem is solved again on the generated mesh and the adaptation process is repeated. The final mesh will be referred to as the quasi-optimal mesh.

There are a few methods for recovering the discrete Hessian H^h from the piecewise linear function u^h defined at mesh nodes. In next sections, we analyze numerically one variational and two projection methods. It was reported in [7] that the projection method was more robust for the Delaunay-like mesh generation algorithm described there. Since we have different experience with the iterative mesh generation algorithm described in the previous section, we decided to analyze the effect of the Hessian recovery onto the interpolation error.

3.1 Variational Methods

Let superelement σ be a set of triangles sharing an interior mesh node a and $H^h(a)$ be the value of the continuous piecewise linear Hessian H^h at this mesh node. The weak definition of the discrete Hessian is as follows:

$$\int_\sigma H^h_{ij}(a)\,\varphi^h_a\,\mathrm{d}x = -\int_\sigma \frac{\partial u^h}{\partial x_i}\frac{\partial \varphi^h_a}{\partial x_j}\,\mathrm{d}x, \tag{4}$$

where φ^h_a is the piecewise linear (P_1) finite element function associated with the node a. Since φ^h_a vanishes on $\partial\sigma$, definition (4) is nothing else but the Green formula.

At a boundary mesh node a, the Hessian $H^h(a)$ is the weighted extrapolation from the neighboring interior nodes [5]:

$$H^h_{ij}(a) = \frac{\sum_b H^h(b)\,m_{ab}}{\sum_b m_{ab}}, \qquad m_{ab} = \int_\sigma \varphi^h_a\,\varphi^h_b\,\mathrm{d}x, \tag{5}$$

where summation goes over mesh points $b \in \partial\sigma$ that are not on $\partial\Omega$.

The approximation properties of the recovery method (4) have been studied in [1, 10]. The local convergence of the discrete Hessian towards the differential one has been established for a class of smooth solutions. Another variational method for the Hessian recovery has been introduced and analyzed in [2].

3.2 Projection Methods

Another group of methods for recovering the Hessian H^h uses the projection technique. The first method from this group reads:

$$H^h = \mathcal{I}_{ZZ}\left(\nabla\left(\mathcal{I}_{ZZ}(\nabla u^h)\right)\right) \tag{6}$$

where $\mathcal{I}_{ZZ}$ is the Zienkiewicz-Zhu (ZZ) projector on the P_1 finite element space [12]. The gradient of u^h is constant inside mesh triangles. To find the gradient of u^h at an interior mesh node a of the superelement σ defined above, we fit (in the least square sense) a linear function to the values of ∇u^h assigned to the centers of mass of triangles in σ. Repeating this procedure twice, we get the piecewise linear Hessian of u^h at the node a. Again, the Hessian is extrapolated to boundary nodes using (5).

In [7], the L_2-projector $\mathcal{I}_{L_2}$ is used instead of the ZZ-projector. However, both projectors do not remove high frequency errors introduced by small non-uniformities in the mesh. The superconvergence properties of the recovered gradient are restored by applying a smoothing operator S^h right after the projection operator:

$$H^h = S^h\mathcal{I}_{ZZ}\left(\nabla\left(S^h\mathcal{I}_{ZZ}(\nabla u^h)\right)\right). \tag{7}$$

As shown in [3], two Jacobi conjugate gradient iterations for the equation $-\Delta v + v = 0$ with the initial guess corresponding to $\mathcal{I}_{ZZ}(\nabla u^h)$ are sufficient to dump high frequency errors in majority of problems. It is pertinent to note that the authors studied only isotropic meshes and the optimal number of Jacobi iteration was found experimentally. In the next section, we shall show that the same conclusion cannot be extended to anisotropic meshes and the maximal norm.

Note that for isotropic solutions, efficient implementation of the L_2-projector can be done by lumping the mass matrix [3]. In other cases, solution of a linear system with a mass matrix is required to find $\mathcal{I}_{L_2}(\nabla u^h)$. Since the mass matrix may be quite stiff on an anisotropic mesh, we think that the $\mathcal{I}_{ZZ}$-projector is a more reasonable choice in (7) than the L_2-projector suggested in [3].

All three Hessian recovery methods have linear complexity with respect to the number of mesh nodes. The variational method is less expensive. The complexity of the second projection method grows linearly with the number of smoothing iterations. The complexity of these methods is less than complexity of the mesh generation algorithm when the initial mesh is far from the optimal one. In the coarse of adaptive iterations the number of mesh modifications decreases and the relative complexity of the Hessian recovery methods grows. The actual numbers are essentially problem dependent.

4 Numerical Experiments

In this section, we analyze numerically three Hessian recovery methods given by (4), (6), and (7). We shall refer to the last two methods as the ZZ-projection and the BX-projection methods, respectively.

In the first example, we consider the problem of minimizing the interpolation error for the function [6]

$$u(x_1, x_2) = \frac{(x_1 - 0.5)^2 - (\sqrt{10}x_2 + 0.2)^2}{((x_1 - 0.5)^2 + (\sqrt{10}x_2 + 0.2)^2)^2}.$$

defined over the unit square $[0, 1]^2$. The function has slight anisotropic singularity at point $(0.5, -0.2/\sqrt{10})$ which is outside the computational domain but close to its boundary. The Hessian of u is the saddle point matrix, i.e. it has one positive and one negative eigenvalue.

The solution isolines and the adapted mesh are shown in Fig. 1. Table 1 shows the L_∞ and H^1 norms of the interpolation error. The maximal norm of the interpolation error is almost reciprocal to N_*. The energy norm is given only for illustration purposes. All the methods require 4 to 5 adaptive iterations to generate quasi-optimal meshes. *The variational method results in the most accurate interpolation solution.*

The gradient smoothing in the BX-projection method does not affect the energy norm of error but increases the maximum norm in 2-3 times. This factor becomes even bigger if we increase the number of smoothing iterations.

Table 1. Experiment 1: L_∞ and H^1 norms of the interpolation error for three Hessian recovery methods.

	Variational method		ZZ-projection method		BX-projection method	
N_*	L_∞	H^1	L_∞	H^1	L_∞	H^1
3000	0.0334	0.206	0.0410	0.211	0.1201	0.234
6000	0.0164	0.146	0.0234	0.147	0.0443	0.158
9000	0.0102	0.119	0.0138	0.119	0.0326	0.123

In the second experiment, we build the optimal mesh for the function proposed in [7]:

$$u(x_1, x_2) = x_2 x_1^2 + x_2^3 + \tanh(10(\sin(5x_2) - 2x_1)).$$

The computational domain is the square $[-1, 1]^2$. The solution is anisotropic along the zigzag curve (see left picture in Fig. 2) and changes sharply in the direction normal to this curve.

Table 2 shows the L_∞ and H^1 norms of the interpolation error. As in the previous example, *the variational method results in the smallest interpolation error.* Similarly, the energy norm of error does not drop down as fast as the

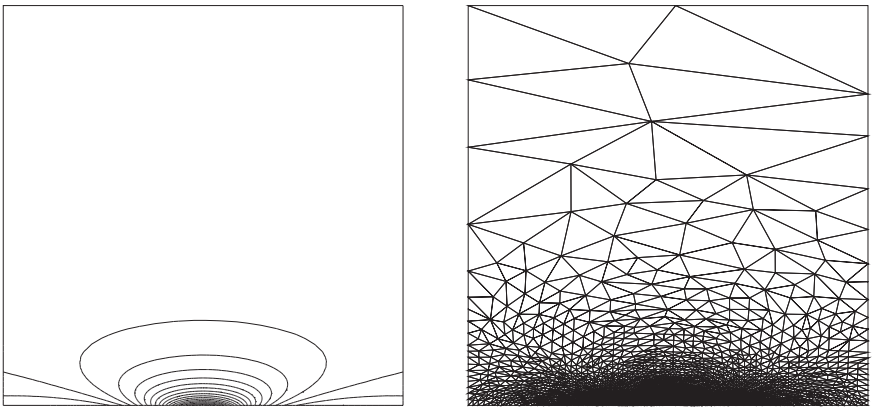

Fig. 1. Isolines of function u from the first example (left) and the adapted mesh after 5 iterations for the Hessian recovered with the variational method (right).

maximal norm since we minimize only the latter. Note that setting the number of smoothing iterations to 10 results in only 19% increase in the energy norm but destroys the anisotropic mesh structure and rockets up the maximal norm.

In Fig. 3, we show the behavior of the maximal norm of error during the course of adaptive iterations. Due to the discrete nature of the metric $|H^h|$, this behavior is frequently non-monotone. We point out the visible oscillations and slower convergence of the BX-projection method. The other two methods required 5 iterations for convergence. We explain this by a slight smearing of the recovered gradient by operator S^h.

Table 2. Experiment 2: L_∞ and H^1 norms of the interpolation error for three Hessian recovery methods.

	Variational method		ZZ-projection method		BX-projection method	
N_*	L_∞	H^1	L_∞	H^1	L_∞	H^1
3000	0.02081	0.920	0.02271	0.945	0.1464	0.906
6000	0.00846	0.686	0.01110	0.889	0.0427	0.716
9000	0.00561	0.650	0.00899	0.627	0.0186	0.635

It is pertinent to note that the L_∞-norm of the error averaged over mesh triangles is close for all three methods. This indicates that the number of lower-quality triangles is much smaller than N_*. However, we should keep it mind that these triangles are usually located in crucial regions from the physical viewpoint.

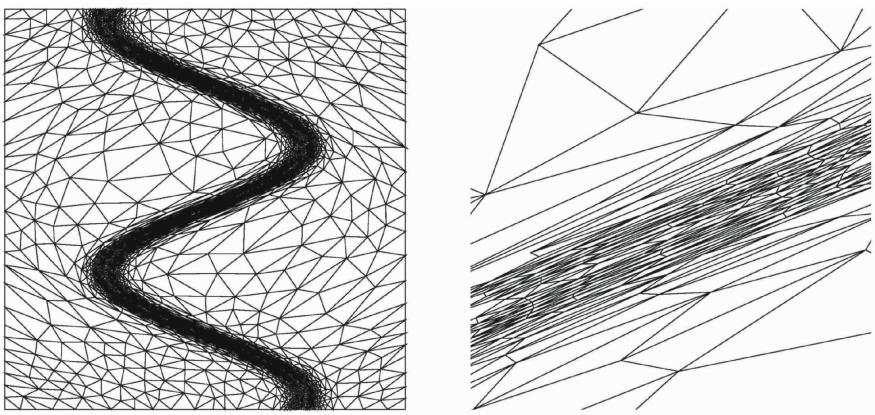

Fig. 2. Adapted mesh after 6 iterations for the Hessian recovered with the variational method (left) and its zoom around point (0, 0) showing the anisotropic mesh structure.

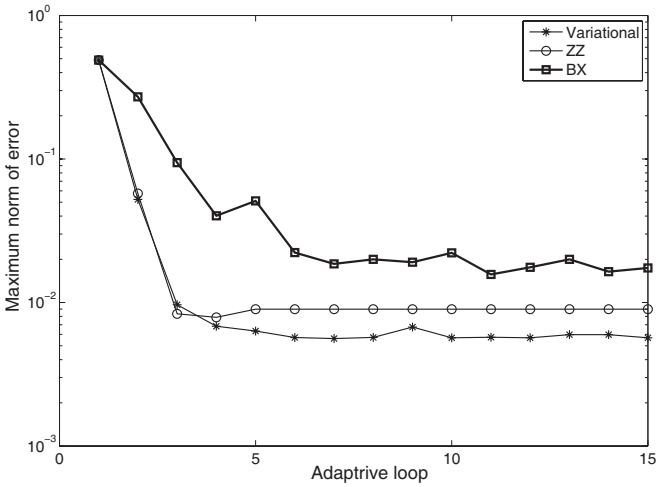

Fig. 3. The L_∞-norm of the interpolation error during the coarse of 15 adaptive iterations with $N_* = 9000$.

In conclusion, we note that the choice of the recovery method must depend on the norm in which we want to minimize the error. For the maximal norm, the gradient smoothing is the bad idea. The variational method seems to exhibit the most robust behavior over a larger scale of norms.

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References

1. Agouzal A, Lipnikov K, Vassilevski Y (1999) Adaptive generation of quasi-optimal tetrahedral meshes, *East-West J. Numer. Math.* **7**, 223–244.
2. Agouzal A, Vassilevski Y (2002) On a discrete Hessian recovery for P_1 finite elements, *J. Numer. Math.* **10**, 1–12.
3. Bank RE, Xu J (2003) Asymptotically exact a posteriori error estimators, part II: General unstructured meshes, *SIAM J. Numer. Anal.* **41**, 2313–2332.
4. Buscaglia GC, Dari EA (1997) Anisotropic mesh optimization and its application in adaptivity, *Inter. J. Numer. Meth. Engrg.* **40**, 4119–4136.
5. Buscaglia GC, Agouzal A, Ramirez P, Dari EA (1998) On Hessian recovery and anisotropic adaptivity, In Proc. of ECCOMAS 98 *the 4th European Computational Fluid Dynamics Conference, 7-11 September 1998, Athensm Greece*, John Wiley & Sons, 1998, Vol. 1, Part 1, pp. 403–407.
6. D'Azevedo E (1991) Optimal triangular mesh generation by coordinate transformation, *SIAM J. Sci. Stat. Comput.* **12**, 755–786.
7. Hecht F (2005) *A few snags in mesh adaptation loops*, In proceedings of the 14th International Meshing Roundtable, Springer-Verlag Berlin Heidelberg.
8. Kunert G (2001) A local problem error estimator for anisotropic tetrahedral finite element meshes, *SIAM J. Numer. Anal.* **39**, 668–689.
9. Picasso M (2003) An anisotropic error indicator based on Zienkiewicz-Zhu error estimator: Application to elliptic and parabolic problems, *SIAM J. Sci. Comput.* **24**, 1328–1355.
10. Vassilevski Y, Lipnikov K (1999) Adaptive algorithm for generation of quasi-optimal meshes, *Comp. Math. Math. Phys.* **39**, 1532–1551.
11. Vassilevski Y, Agouzal A (2005) An unified asymptotical analysis of interpolation errors for optimal meshes, *Doklady Mathematics* **72**, 879–882.
12. Zhu JZ, Zienkiewicz OC (1990) Superconvergence recovery technique and a posteriori error estimators, *Inter. J. Numer. Meth. Engrg.* **30**, 1321–1339.