Optimal Mesh for P_1 Interpolation in H^1 Seminorm

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Summary. In this paper we present one approach to build optimal meshes for P_1 interpolation. Considering classical geometric error estimates based on the Hessian matrix of a solution, we show it is possible to generate optimal meshes in H^1 seminorm via a simple minimization procedure.

1 Introduction

1.1 Statement of the Problem

Let Ω be a domain of $I\!R^2$ and u a solution of an elliptic PDE problem on Ω under Dirichlet boundary conditions on $\partial\Omega$.

The energy norm is the natural one to measure the error of a numerical approximation of the solution of this problem and the Cea's lemma [4] give a bound for the error.

Cea's lemma. If the hypotheses of the Lax-Milgram theorem are satisfied, there exists a constant C independent of the subspaces V_h such that

$$\|u - u_h\| < C \inf_{v \in V_h} \|u - v\|$$

where V_h denotes the finite element spaces, u_h the discrete solution associated with V_h and $\|.\|$ the norm in the space V.

a posteriori mesh adaptation. Thanks to Cea's lemma, the error made over the mesh is bounded by the interpolation error. Hence, controlling the interpolation error allows control the approximation error. This has led to a number of a posteriori mesh adaptation procedures with metric specifications based on the control of the interpolation error ([9, 7] and the references in [13]). However, these methods are usually based on the L^∞ norm of the interpolation error instead of a more physical norm.

The exact solution is usually unknown; we only know the discrete solution u_h and reconstruction methods are used to estimate the error estimator since it is theoretically bounded by derivatives one order higher than the order of the field approximation.

Surface mesh optimization. If we consider the solution u as a Cartesian surface, the problem of minimizing the interpolation error consists in defining a mesh so that the surface mesh is as smooth as possible. This has led to the notion of geometric mesh [9] : a geometric mesh is a piecewise planar approximation of the surface so that the distance between the underlying surface and the element of the triangulation does not exceed given tolerance threshold. To make sure that a mesh is a geometric mesh, one generally introduces some measurements of quality like planarity, roughness or deviation. In particular the roughness of a piecewise linear surface is defined as the square L^2 norm of the gradient of the surface integrated on the triangulation [15]. Thus, controlling the H^1 seminorm can be see as an optimization procedure of the quality of a surface.

In this paper, we assume that the discrete solution u_h is a P_1 continuous finite element function and that we measure the error in H^1 seminorm. We shall see that the computation of this error involves second order derivatives of the computed scalar field. One way estimate these derivatives is to use reconstruction methods (see [14] for a comparison of recently published recovery methods).

1.2 Outline

The purpose of this paper is to describe an adaptation procedure in order to build an optimal mesh by controlling the H^1 seminorm of the interpolation error. First we go back to the H^1 seminorm of the interpolation error on one element of the mesh. We show how it can be written using the Hessian matrix of u and we recall how the second order derivatives of the computed scalar field can be estimated over the domain using recovery methods. Then an adaptation procedure based on a minimization problem is described. Eventually, analytical examples are provided to illustrate this approach.

2 Optimal Mesh

The concept of optimal mesh theoretically refers to a set of meshes. For example, if several meshes are given, one will be able to say that the optimal mesh is that for which the selected measurement is optimal. In this work, a triangulation consisting of N nodes will be said optimal in H^1 seminorm if it is the triangulation that minimizes the interpolation error among all conforming N-nodes triangulations.

3 Expression of the H^1 Seminorm of the P_1 Interpolation Error

We consider a mesh $\mathcal{T}(\Omega)$ composed of linear triangles and denote $K = [a, b, c]$ as the reference element, $\Pi_h u$ as the piecewise linear interpolate of the solution u according to $\mathcal{T}(\Omega)$. The usual $\mathbb{R}^2$ scalar product is denoted by $\langle ., . \rangle$, and the Hessian matrix of u by H_u.

3.1 Expression of the Error Gradient on One Element K

Let us write $e_K = u - \Pi_h u$ the interpolation error on element K. It vanishes at the vertices of the triangle and has the same Hessian matrix as u.

Using a Taylor expansion at each vertex v_i of K (where the error vanishes) we can write,

$$e_K(v_i) = e_K(x) + \langle \overrightarrow{v_i x}, \nabla(e_K)(x) \rangle + \frac{1}{2} \langle \overrightarrow{v_i x}, H_u \overrightarrow{v_i x} \rangle + \varepsilon_i \|\overrightarrow{v_i x}\|^2, \forall x \in K, \quad (1)$$

Let us denote $q_{v_i}(x) = \frac{1}{2} \langle \overrightarrow{v_i x}, H_u(x) \overrightarrow{v_i x} \rangle$.

Then, by neglecting the terms of ε_i, we have

$$\begin{cases} \langle \overrightarrow{v_0 x}, \nabla(e_K)(x) \rangle + q_{v_0}(x) = 0 \\ \\ \langle \overrightarrow{v_1 x}, \nabla(e_K)(x) \rangle + q_{v_1}(x) = 0 \\ \\ \langle \overrightarrow{v_2 x}, \nabla(e_K)(x) \rangle + q_{v_2}(x) = 0 \end{cases}$$

and consequently the following system

$$\begin{cases} \langle \overrightarrow{v_0 v_1}, \nabla(e_K)(x) \rangle + q_{v_0}(x) - q_{v_1}(x) = 0 \\ \\ \langle \overrightarrow{v_0 v_2}, \nabla(e_K)(x) \rangle + q_{v_0}(x) - q_{v_2}(x) = 0 \end{cases}$$

which can be written

$$A \nabla(e_K)(x) = -F(x) \quad (2)$$

where $A = (\overrightarrow{v_0 v_1} \ \overrightarrow{v_0 v_2})^t$ and $F(x) = \begin{pmatrix} q_{v_0}(x) - q_{v_1}(x) \\ q_{v_0}(x) - q_{v_2}(x) \end{pmatrix}$.

3.2 Seminorm

The H^1 seminorm of the interpolation error on an element K is

$$|e_K|^2_{H_1} = \int_K \|\nabla(e_K)(x)\|^2.$$

Finally, with the approximation (2) we consider

$$|e_K|^2_{H_1} \approx \int_K \|A^{-1} F(x)\|^2. \quad (3)$$

3.3 Remark on the Numerical Computation of the Error Gradient

To compute the error gradient we need to get an estimate of the Hessian matrix on K. Integrals of the error gradient can easily be obtained via a quadrature formula.

Estimation of the Hessian The Hessian of u_h, a P_1 finite element function, can be approximated by using a recovery method. We can cite for example the generalized finite difference (a variant of Green's formula [10, 10.4.2]), the Zienkiewicz-Zhu [18] recovery procedure, the simple linear fitting [6], the quadratic fitting [17] or the double L^2 projection

$$H = I_{L^2}(\nabla(I_{L^2}(\nabla u))) \tag{4}$$

where I_{L^2} is the L^2 projection on the P^1 finite element space [1].

We refer to [14] for a comparison of these different recovery techniques.

Moreover, the adaptation process may be controlled by modifying the eigenvalues in the spectral decomposition of H_u (see [16] for an analysis of the effect of such a control strategy on the interpolation error).

Here, to evaluate the Hessian, we use formula (4). As it has been said in [13], we have no convergence proof of this scheme but result is better with using it [1].

Interpolation scheme In order to get the value of the Hessian everywhere on the mesh, interpolation schemes are used since the Hessian is only known at each vertex of the mesh.

4 Application to Mesh Adaptation

The procedure described below minimizes the interpolation error among all triangulations having the same number of vertices.

Following [8], we introduce a functional on the mesh based on the error gradient to drive the adaptation procedure. Hence, we define

$$\mathcal{J}(\mathcal{T}) = \sum_{K \in \mathcal{T}} \int_K ||\nabla(e_K)(x)||^2. \tag{5}$$

Using (3) we can write the expression of $\mathcal{J}$ with the Hessian matrix of u and we consider now

$$\mathcal{J}(\mathcal{T}) = \sum_{K \in \mathcal{T}} \int_K ||A^{-1}F(x)||^2. \tag{6}$$

The procedure consists in minimizing functional $\mathcal{J}$ using local operators on the initial mesh.

4.1 Operators for Mesh Improvement

There are four main types of mesh improvement, topologic or geometric: edge refinement, edge swapping, vertex suppression, vertex displacement [10]. Since we want to adapt the mesh while preserving the same number of vertices, we only consider node reconnection and vertex displacement.

Edge swapping. It is a simple operation in 2D. Only two configurations are possible as an edge has only two adjacent triangles. The edge between two triangles $K1$ et $K2$ will be swapped if:

$$\int_{K_1} ||A^{-1}F(x)||^2 + \int_{K_2} ||A^{-1}F(x)||^2 > \int_{K_1'} ||A^{-1}F(x)||^2 + \int_{K_2'} ||A^{-1}F(x)||^2$$

where ($K1'$,$K2'$) is the alternate configuration.

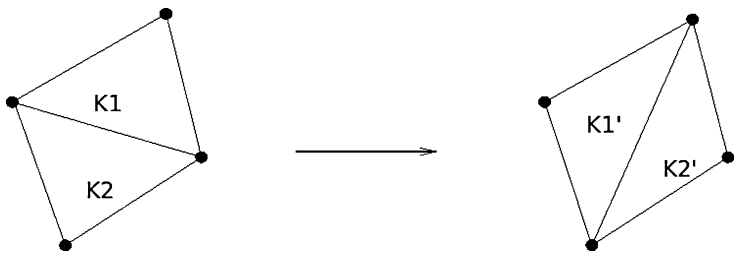

Fig. 1. *swap*

Vertex displacement. This procedure preserves the connectivity of the mesh. In order to get the position of the nodes that minimises the error we use a minimisation procedure.

Let $V = (v_1, v_2, ..., v_N) \in I\!R^{2^N}$ where N is the number of the vertices of $\mathcal{T}$.

Let us consider

$$\mathcal{F} : I\!R^{2^N} \to I\!R,\ V \to \mathcal{F}(V) = \sum_{K \in \mathcal{T}} \int_K ||A^{-1}F(x)||^2$$

and the following optimization problem

$$\min_{v \in I\!R^{2^N}} \mathcal{F}(v),$$

for which classical optimization methods [3] can be used. We use a gradient method to minimize $\mathcal{J}$:

$$\forall i \in \{0, .., N\} \quad v_i^{n+1} = v_i^n - \rho(\nabla \mathcal{J}(V^n))_i, \quad \rho > 0.$$

The main difficulty of this procedure is to move the points without creating overlapping elements. To ensure that the movement does not destroy a valid triangulation, we adapt the step size ρ so that the new point $x^\star$ stays in the shell around x (i.e. all the elements around x).

4.2 Algorithm

The algorithm consists in applying iteratively these operators on the mesh until the mesh stabilizes.

- We start with a triangulation $\mathcal{T}_h^0$ of the computational domain.
- Repeat:
  - Compute the discrete Hessian at each vertex of the mesh $\mathcal{T}_h^{(i)}$
  - Generate the mesh $\mathcal{T}_h^{(i+1)}$ which minimize (5) with the following local operators:
    - Vertex displacement
    - Edge swapping
  - Terminate the loop if the number of swaps is null and if the vertices don't move.

Remarks:

Since the edge swapping is performed if and only if the functional (6) decrease, the number of swap is bounded.

5 Experimental Results

To illustrate the proposed method, we present some results for a few examples : three examples with given analytical solutions and one for a partial differential equation.

Here the goal is to generate optimal triangulation of these surfaces by using the adaptation scheme defined previously. The resulting triangulation is considered optimal in so far as it is the triangulation that minimimizes the interpolation error in H^1 seminorm among all the surface triangulations (and ensure the best error equirepartion). As we want compare our procedure with classical adaptation schemes, we start with a mesh of the parameter space (we obtain the surface mesh by considering the value of the analytical function at each mesh vertex) obtained from a size map built by considering the absolute value of the Hessian matrix. More precisely the initial mesh is built by using a metric based adaptation procedure which equilibrates the interpolation error in L^∞ norm over the mesh and the metric field is computed using the absolute value of the Hessian matrix.

For each example we present the value of the functional before and after optimization. To study the convergence of the interpolation error as the mesh is refined, we plot the H^1 seminorm of the error as functions of the number of elements nbt. In all these examples, for technical reason, all the boundary points are fixed.

5.1 Analytical Examples

We consider an initial mesh (built with the FreeFem++ software [12]) for which the interpolation error in L^∞ norm is upperbounded by a threshold value ε.

Constant Hessian surfaces We consider here adaptive meshes for $f_1(x, y) = x^2 + y^2$ and $f_2(x, y) = x^2 - y^2$ (Fig. 2).

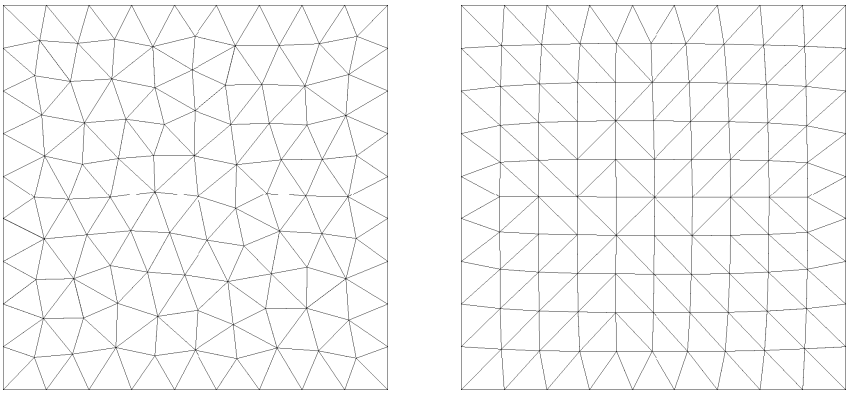

Fig. 2. Resulting meshes in the parameter space for two different types of surfaces: $f(x, y) = x^2 + y^2$. (left) and $f(x, y) = x^2 - y^2$. (center), $(x, y) \in [-1, 1] \times [-1, 1]$, where the triangles are right-angled and aligned along the axes.

Non constant Hessian surfaces We consider here adaptive meshes $f_3(x, y) = x^3 + y^3$ (Fig. 3) and $f_4(x, y) = x^2y + y^3 + \tanh(10(\sin(5y) - 2x))$ (Fig. 5) which has a more complicated structure and exhibits an highly anisotropic feature (this function simulates a solution with a shock layer : the equation of the smooth line is $sin(5x) = 2x$)

Both procedures equilibrate the interpolation error. Nevertheless, if we compare the initial mesh to the final one, the pattern of the triangles is different between the elliptic and the hyperbolic regions. We obtain the same triangulation in the elliptic region (where the Hessian matrix is positive definite) but, as expected not in the hyperbolic region: for quadratics functions, the optimal triangles which produce the smallest H^1 seminorm of the interpolation error are acute isoceles triangles aligned with the solution [5].

We observe the same behaviour for more general functions (see the example Fig. 5).

However, the value of the functional in the case of the adaptation in L^∞ norm and in our case are close to each other for smooth functions (see Fig. 4) especially for functions whose adapted mesh has little anisotropy: in this case,

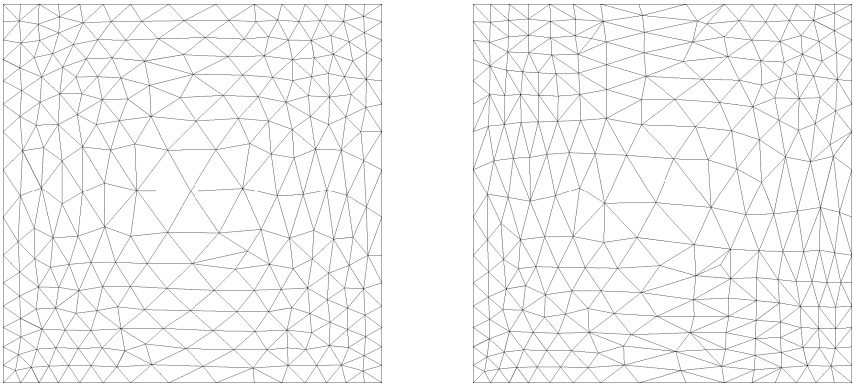

Fig. 3. Initial (left) and resulting (right) mesh in the parameter space for $f_3(x, y) = x^3 + y^3$,$(x, y) \in [-1, 1] \times [-1, 1]$

the classical metric based adaption technique gives good meshes even in the H^1 seminorm. This is not surprising in so far as that in this case the corresponding mesh is close to a Delaunay mesh for which roughness (and consequently the cost function $\mathcal{J}$) is minimal [15].

5.2 Partial Differential Equation Examples

This example solves the boundary value problem of Poisson's equation on an L-shape with Dirichlet boundary condition:

$$-\Delta u = 1, \Omega =]0, 1[^2 - [1/2, 1[^2$$

Numerical results are shown in Table 1. Similar observations as in the previous analytical cases can be made. For a PDE with shock the key point is the reconstruction of the Hessian which can be quite difficult on highly anisotropic case [13].

Table 1. Value of the H^1 norm and L^2 norm for initial (right) and resulting mesh (left) as function of the number of elements(nbt) or vertices (nbv)

H1	L2	H1	L2	nbv	nbt
0.0409041	0.00225554	0.0426227	0.00234673	57	84
0.0129854	0.000262934	0.0136801	0.000255626	271	473
0.00906429	0.000128142	0.00944111	0.000135652	368	659
0.00633305	6.43262e-05	0.0065542	6.88933e-05	684	1264
0.00470695	3.71549e-05	0.00493037	4.07533e-05	1141	2144

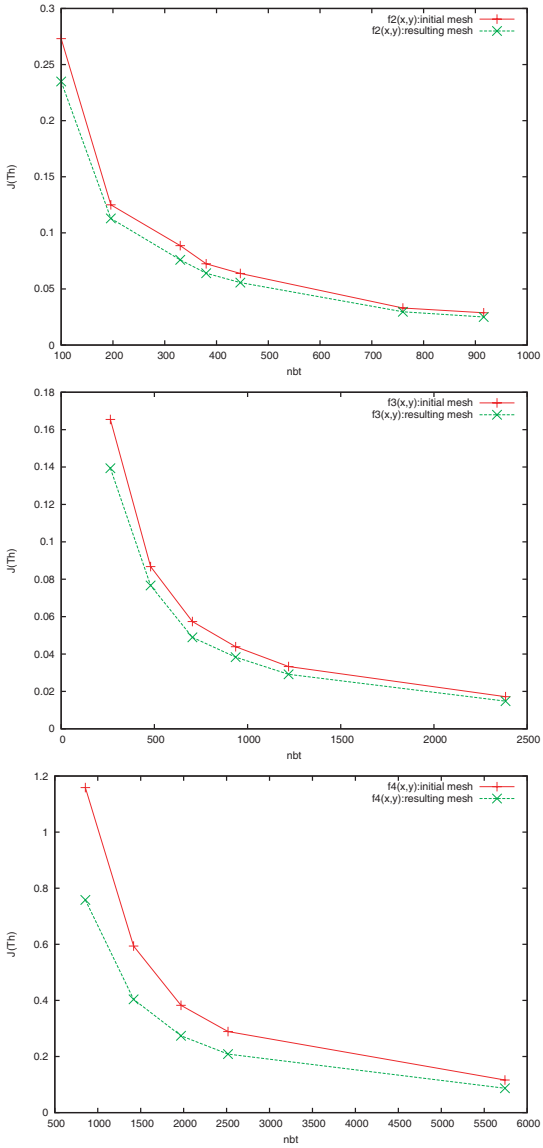

Fig. 4. Plot of $\mathcal{J}(Th)$ as functions of nbt for $f_2(x,y)$, $f_3(x,y)$ et $f_4(x,y)$

5.3 Towards Quad-Triangles Meshes

Moreover, as for saddle-shaped functions, quadrilaterals may offer a higher-order approximation on a mesh [11], this procedure can be used for building (anisotropic) quadrilaterals in the hyperbolic region through a simple triangle-to-quad conversion [2].

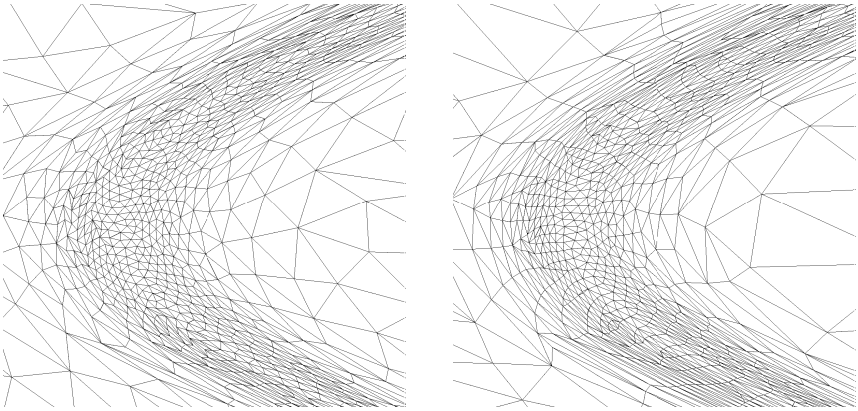

Fig. 5. Zoom of meshes of $f_4(x, y) = x^2y + y^3 + \tanh(10(\sin(5y) - 2x))$ in the parameter space (initial mesh on the left and final mesh on the right), $(x, y) \in [-1, 1] \times [-1, 1]$, $x^2 + y^2 < 1$.

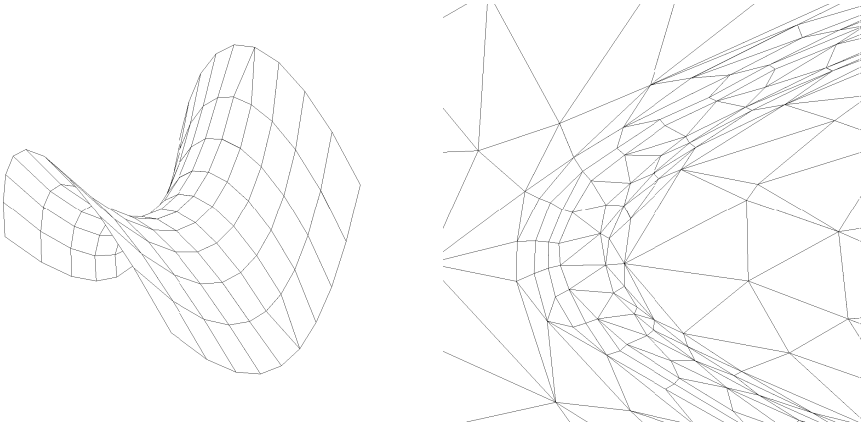

Fig. 6. Resulting surface meshes for $f_2(x, y)$ (left) and $f_4(x, y)$ (rigth) where triangulated mesh has been changed into a quad-dominant mesh in hyperbolic region through a simple triangle-to quad conversion.

6 Conclusion

In this paper we have presented a simple methodology based on local mesh operators for building optimal meshes for the H^1 seminorm of the interpolation error. Numerical results were presented to illustrate our approach.

- We have shown through numerical experiments that classical methods based on the absolute value of the Hessian matrix of the solution allow the construction of good meshes even for the H^1 seminorm, especially from the finite element viewpoint.

- From a surface meshing viewpoint il would be interesting to study in detail the optimization in H^1 seminorm especially for building quad-dominant meshes in anisotropic regions.

This method can be extended to 3D. In this case the local operators for mesh modifications are much more complex : the edge swap operator modifies the shell of an edge (ie all the tetrahedra around an edge) and the number of possibly new tetrahedralizations of a shell of n elements is given by the Catalan number $C(n) = \frac{(2n-2)!}{n!(n-1)!}$ which shows the complexity of this operator.

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