Geometric Dissections
Now Swing and Twist

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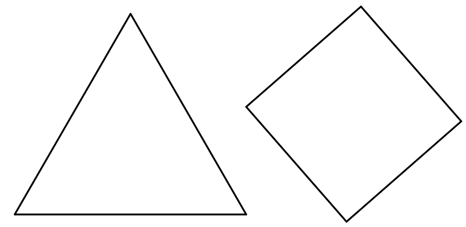
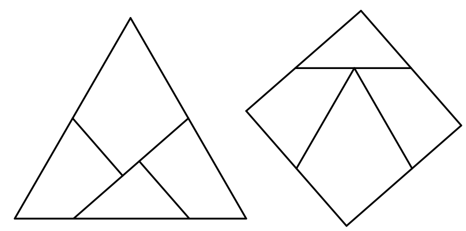
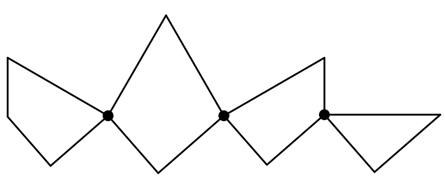
An equilateral triangle to a square:



Dissection (4 pieces)
[Henry Dudeney (or Charles McElroy?) 1902]:

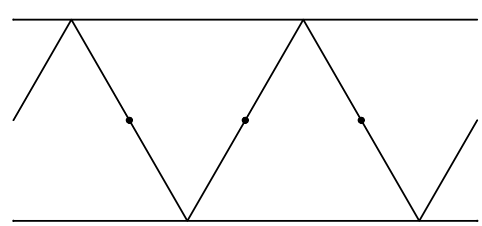


Swing-hinged pieces: triangle to a square
[Henry Dudeney 1907]:

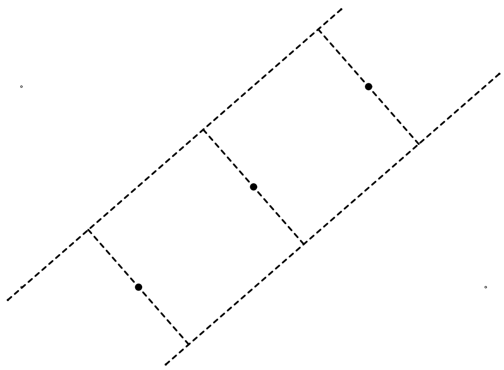


“I add an illustration showing the puzzle in a rather curious practical form, as it was made in polished mahogany with brass hinges for use by certain audiences. It will be seen that the four pieces form a sort of chain, and that when they are closed up in one direction they form the triangle, and when closed in the other direction they form the square.”

Strip of equilateral triangles



+
Strip of squares



Outline

Introduction

Swing-hinged dissections from

- Tessellations
- T-Strips
- Completing the tessellation
- Polygon structure

Twist-hinged dissections from

- Converting swing hinges
- Parallelogram twist
- Completing the pseudo-tessellation

Conclusion

Some History

Standard Dissections

Plato	4th cent., BCE
Thābit	9th cent., CE
Abū'l-Wafā	10th cent.
Anon. (Abū Bakr ?)	ca. 1300
Leonardo da Vinci	ca. 1500
Cardano	1557
Tai Chen	18th cent.
Montucla	1778
John Jackson	1821
...	

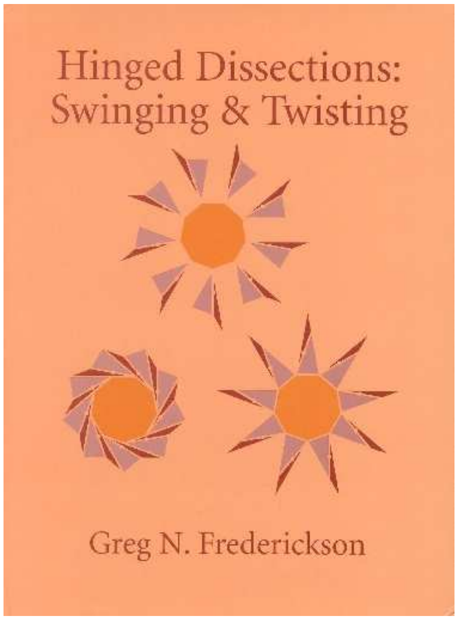
Swing-Hinged Dissections

Philip Kelland	1864
? Henry Taylor	1905
Henry Dudeney	1907
Robert Yates	1949
Harry Lindgren	1960
...	
Akiyama + Nakamura	1998, 2000
GNF	1997–2000

Twist-Hinged Dissections

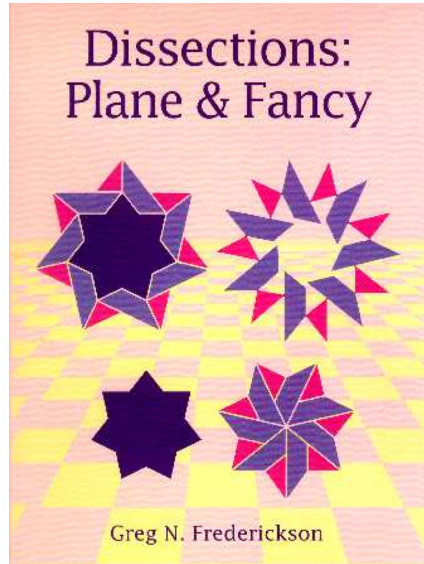
Erno Rubik	1983
E. Lurker, Wm. Esser	1984, 1985
GNF	1999–2000

Based on my recent book (2002):



http://www.cs.purdue.edu/homes/gnf/book2.html

Also see my first book (1997):

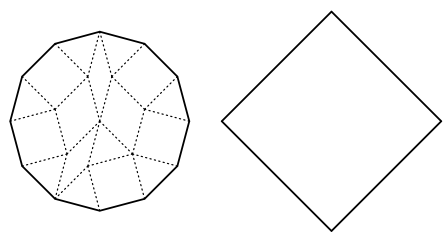


http://www.cs.purdue.edu/homes/gnf/book.html

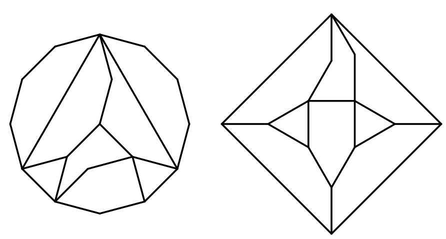
Swing Hinged Dissections From

Superposing Tessellations

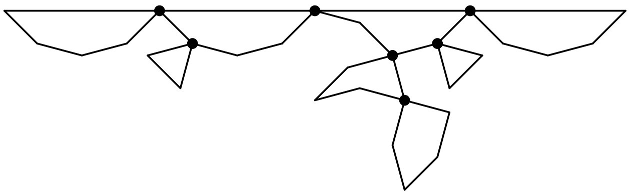
A regular dodecagon to a square:



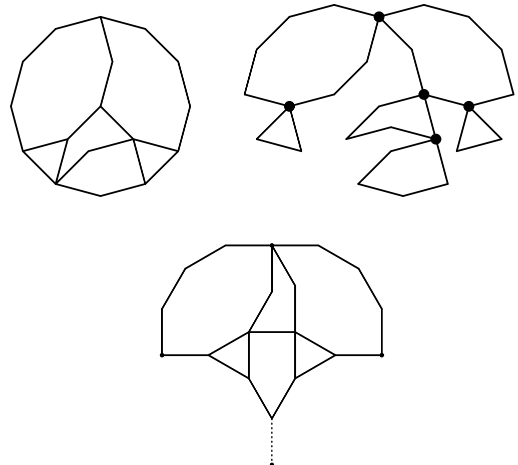
Swing-hingeable dissection (8 pieces) [GNF, 1997]:



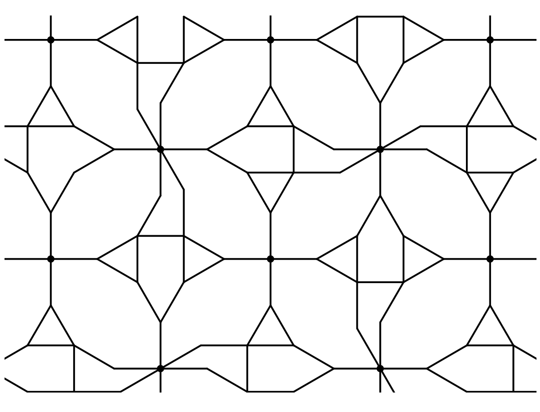
Swing-hinged pieces: dodecagon to square



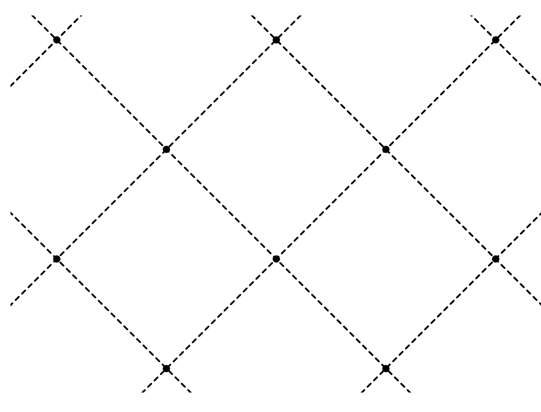
Creating a hinged tessellation element for a dodecagon:



Tessellation of dodecagons



+
Tessellation of squares



A *symmetry point* of a tessellation is a point about which there is rotational symmetry.

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be superposed so that points of intersection between line segments are at symmetry points.

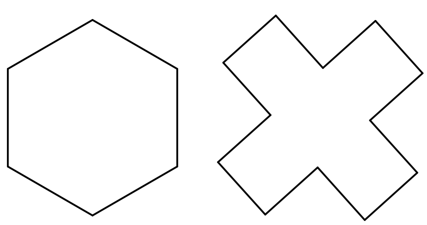
The superposition is *proper intersecting* if $\mathcal{T}_1$ and $\mathcal{T}_2$ share no line segments of positive length.

Theorem. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two tessellations of hinged elements that have a superposition that is proper intersecting. Then the induced dissection is hingeable.

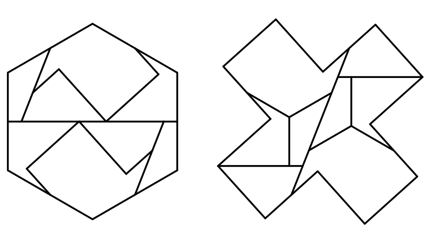
Swing Hinged Dissections From

Crossposing T-Strips

A regular hexagon to a Greek cross:

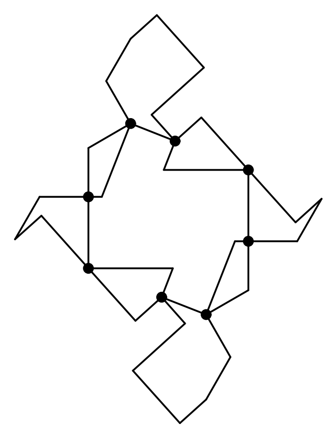


Swing-hingeable dissection (8 pieces) [GNF, 1999]:



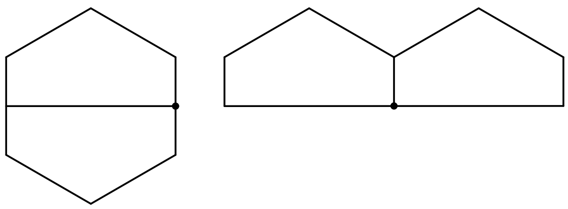
Rotational Symmetry

Swing-hinged pieces: hexagon to cross

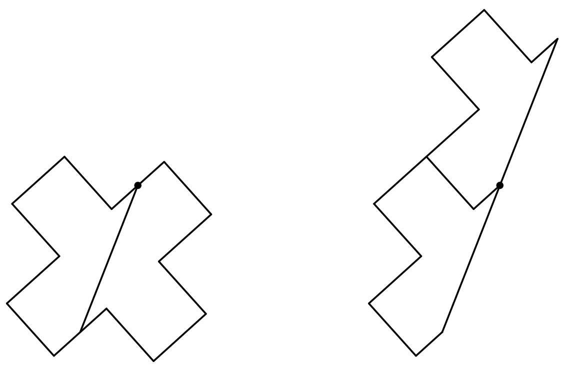


Cyclicly hinged

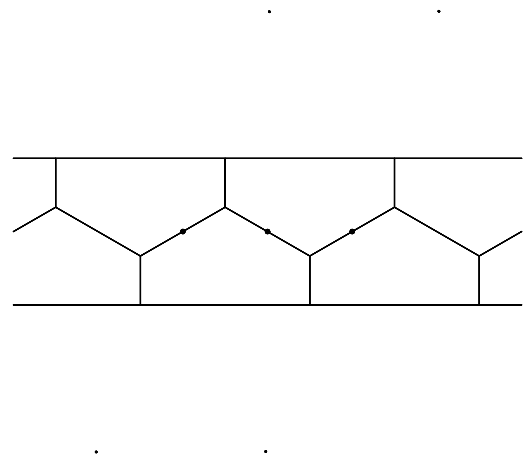
Twinned strip element for a hexagon:



Twinned strip element for a Greek cross:

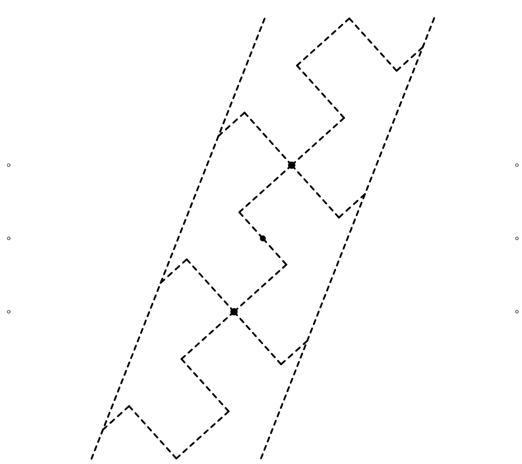


Strip for hexagons



+

Strip for Greek crosses



An *anchor point* is a point of 2-fold rotational symmetry shared by two consecutive elements in the T-strip.

Theorem. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two strips of hinged elements. If $\mathcal{S}_1$ and $\mathcal{S}_2$ are crossposed so that points of intersection between line segments are where

two anchor points coincide,

an anchor point falls on a strip boundary,

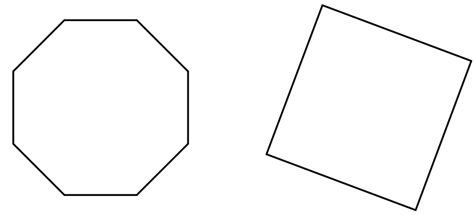
or two strip boundaries cross,

then the induced dissection is hingeable.

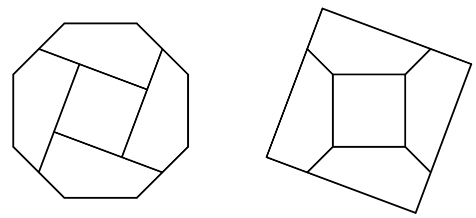
Swing Hinged Dissections From

Completing the Tessellation

A regular octagon to a square:

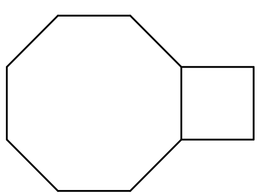


Dissection (5 pieces)
[Anonymous (Abū Bakr ?), ca. 1300]:

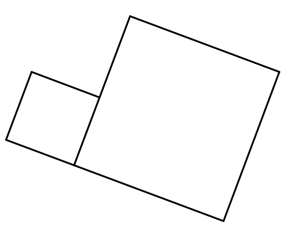


Rotational symmetry

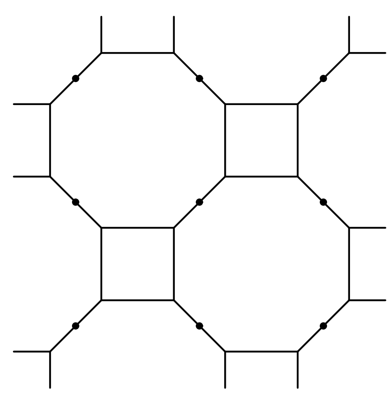
Completing a tessellation element
for an octagon:



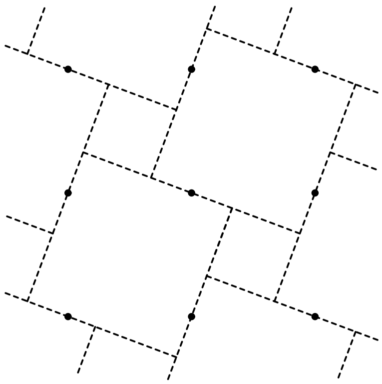
and for a square:



Tessellation of octagons and little squares

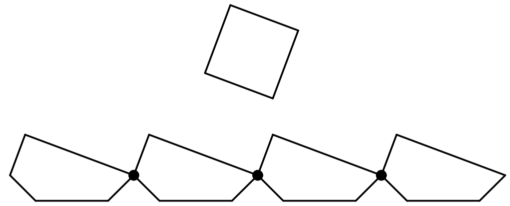


+
Tessellation of big and little squares



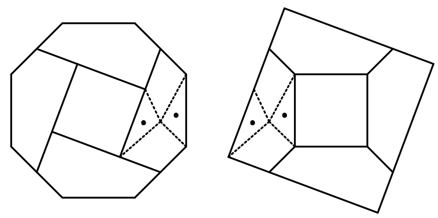
24-a

Hinged pieces: octagon to a square . . .



But – OOPS – not all hinged!

Split one piece to get interchangeable pieces:



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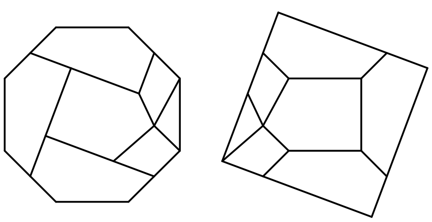
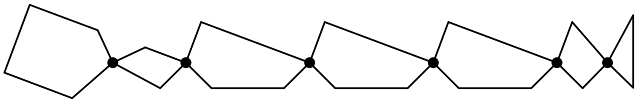
Swing-hingeable dissection (7 pieces)
[GNF, 1999]:



Swing-hinged pieces: octagon to a square



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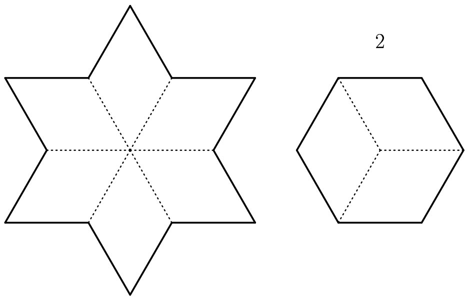
Swing Hinged Dissections From

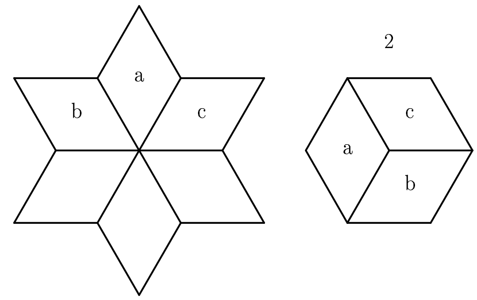
Polygon Structure

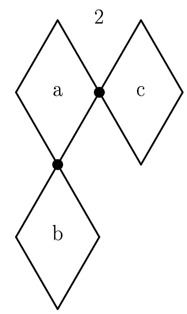
27

A hexagram to two regular hexagons:



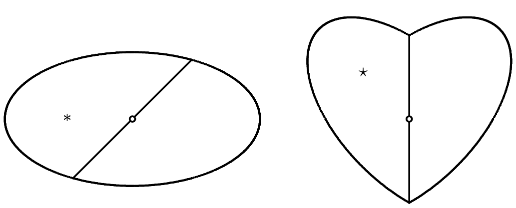
Swing-hinged dissection (6 pieces) [GNF, 1999]:





Twist Hinges – An Example

Ellipse to a heart
[William Esser, III, 1985]:
(similar to Ernst Lurker, 1984)



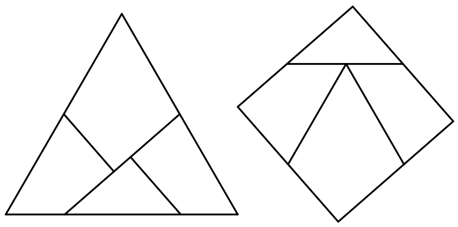
twist hinge

- on the interior of a shared edge
- rotation perpendicular to the edge

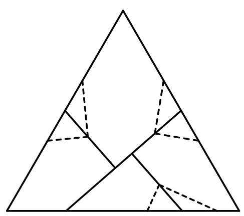
Twist Hinged Dissections from

Converting Swing Hinges

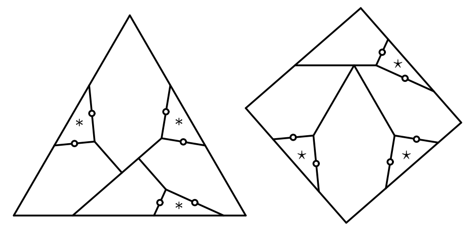
Return to equilateral triangle to square:



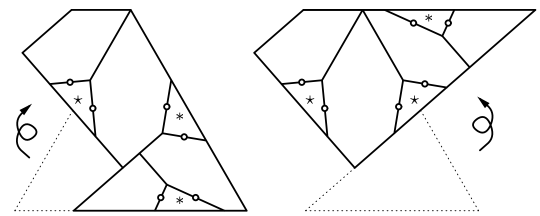
Use isosceles triangles at hinge points:



Twist-hingeable dissection (7 pieces) [GNF, 1999]:



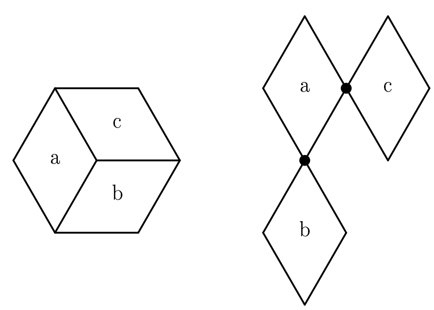
Twisting: intermediate configurations



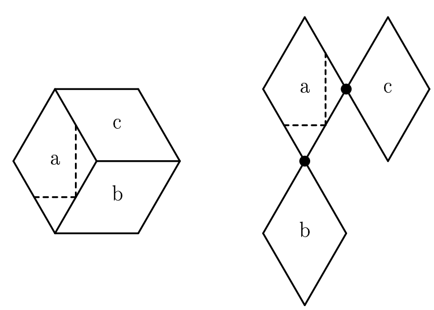
Two pieces that are connected by a swing hinge are *hinge-snug* if they are adjacent along different line segments in each of the figures formed, and each such line segment has one endpoint at the hinge.

Theorem. Let $\mathcal{D}$ be a swing-hingeable dissection such that each pair of pieces connected by a hinge is hinge-snug. We can then replace each swing hinge with a new piece and two twist hinges, so that the resulting dissection $\mathcal{D}'$ is twist-hingeable.

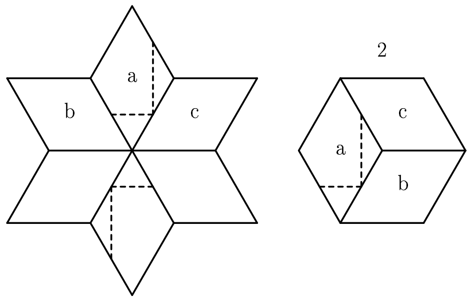
Return to a hexagram to two hexagons:



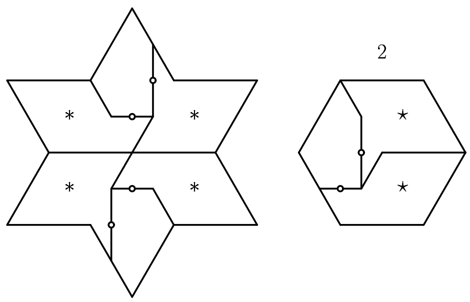
Use an isosceles triangle at each hinge point:



Copy the isosceles triangles:



Twist-hingeable (6 pieces) [GNF, 1999]:



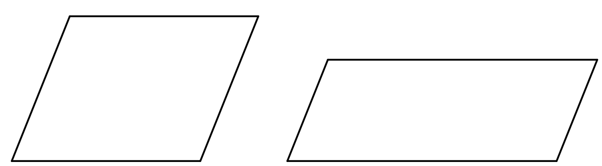
A hinged assemblage is *hinge-reflective* if when we flip all pieces in this hinged assemblage on to their other side, then there is no effective change to the whole hinged assemblage.

Theorem. Let hinged dissection $\mathcal{D}$ have two hinge-snug pieces, such that the hinged assemblage on one side of the swing hinge is hinge-reflective. Then we can modify the two pieces and replace the swing hinge with a twist hinge.

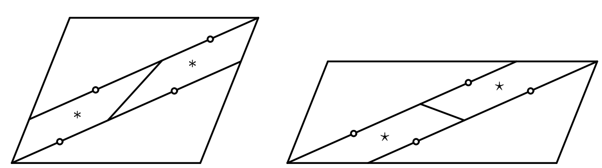
Twist Hinged Dissections From

Parallelogram Twist

Change length of parallelogram:



Twist-hingeable Dissection (4 pieces)
[GNF, 1999]:



Theorem. The P-twist can convert a parallelogram with sides a and $b \leq a$ and nonacute angle θ to any parallelogram with the same nonacute angle and a side from a up to, but not including, $a + \sqrt{a^2 + b^2 - 2ab\cos\theta}$.

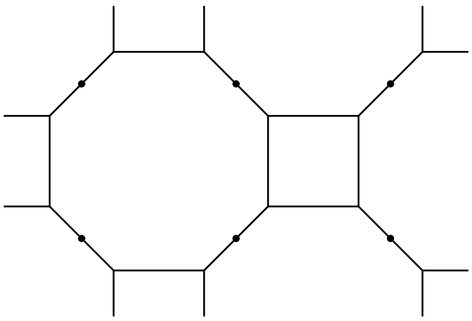
In particular, the P-twist works for rectangles. ($\theta = 90°$)

Twist Hinged Dissections From

Completing the Pseudo-Tessellation

Return to octagon to square:

Tessellation of octagons and little squares



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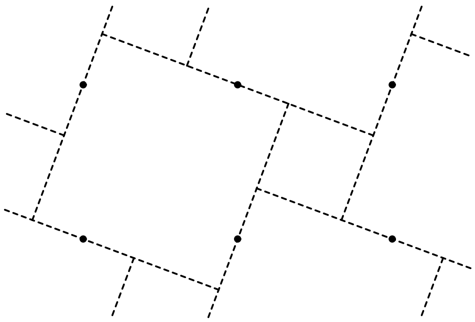
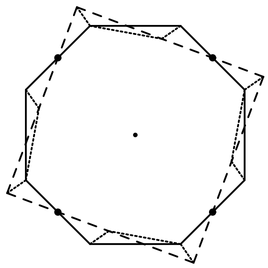
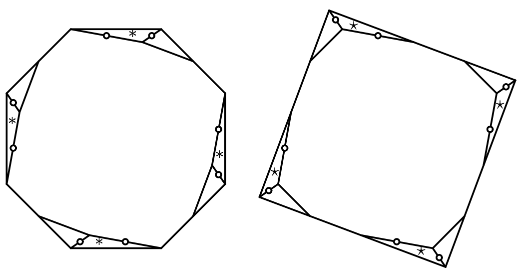
Tessellation of big and little squares



Overlaying octagon and square:



Twist-hinged dissection (9 pieces)
[GNF, 1999]:



Let $\{p\}$ be a regular polygon with p sides.

Theorem. Completing the pseudo-tessellation gives a $(2p+1)$-piece twist-hingeable dissection of a $\{2p\}$ to a $\{p\}$.

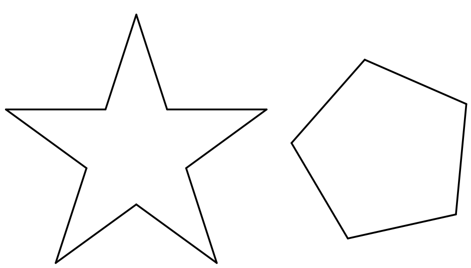
hexagon to triangle

octagon to square

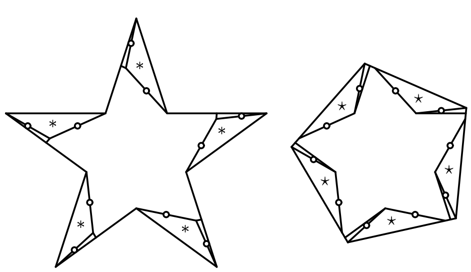
decagon to pentagon

...

A pentagram to a pentagon



Twist-hinged dissection (11 pieces)
[GNF, 1999]:



Let $\{p/q\}$ be a star with p points (vertices), where each point is connected to the q-th points clockwise and counterclockwise from it.

Theorem. Completing the pseudo-tessellation gives a $(2p+1)$-piece twist-hingeable dissection of a $\{p/q\}$ to a $\{p\}$ whenever $p \geq 3q - 1$.

$\{5/2\}$ to pentagon

$\{6/2\}$ to hexagon

$\{7/2\}$ to heptagon

$\{8/2\}$ to octagon

$\{8/3\}$ to octagon

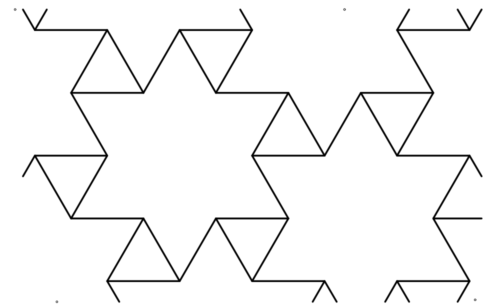
...

Further Example of

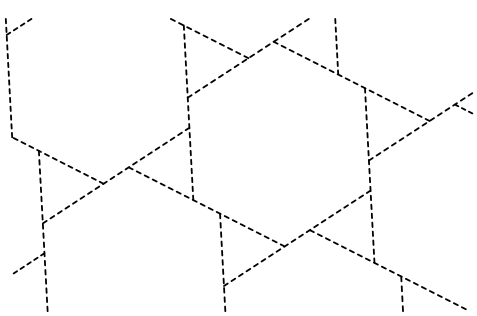
Twist Hinged Dissections

Improve hexagram to hexagon:

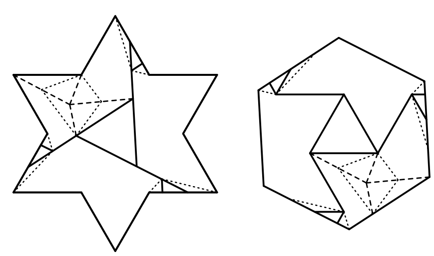
Tessellation of hexagrams and triangles



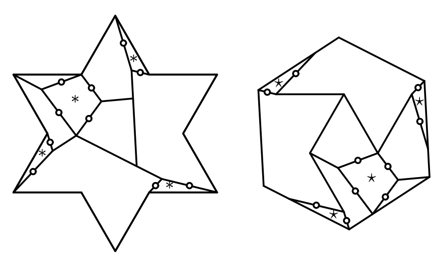
+
Tessellation of hexagons and triangles



Add twists onto a hexagram to a hexagon:



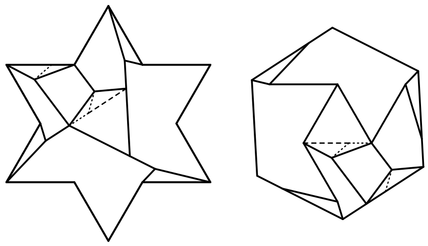
Twist-hinged dissection (10 pieces)
[GNF, 2000]:



Eight of the pieces are cyclicly hinged

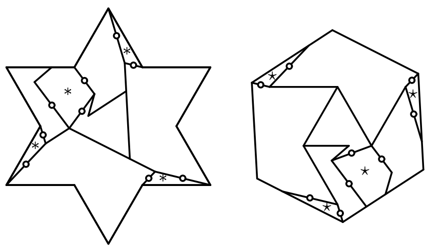
A surprise by Gavin Theobald!

– one more isosceles triangle:



Twist-hinged dissection (9 pieces)

[Gavin Theobald, 2002]:



General Thoughts

Generality Issues

For any two figures of equal area
and bounded by straight line segments:

Is a dissection possible?

Yes. [Wallace, 1831], [Bolyai, 1832], [Gerwien, 1833]

Is a swing-hingeable dissection possible?

To mirror image. [Eppstein, 2001]

General case – OPEN

Is a twist-hingeable dissection possible?

– OPEN

Hardness Issues

Theorem. (Hearn, Demaine, Frederickson)

Given a dissection, a hinging, and two convex hinged configurations, it is PSPACE-hard to determine whether it is possible to move from one configuration to the other.

(By reduction from the PSPACE-complete problem of nondeterministic constraint logic)

Conclusion

Hinged dissections:

- explore interaction of geometry + motion
- give insight into symmetry + tessellations
- synthesize aspects of CS, MATH, + ME
- provide enrichment in math education
- are lots of fun!

What's Next?

Piano-Hinged Dissections: Time to Fold

completed manuscript,

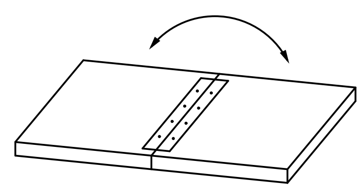
320 pages, August 2004.

Piano Hinges

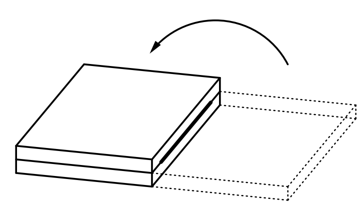
– A third type of hinge

To use piano hinges,
"2-D" dissections need two levels:

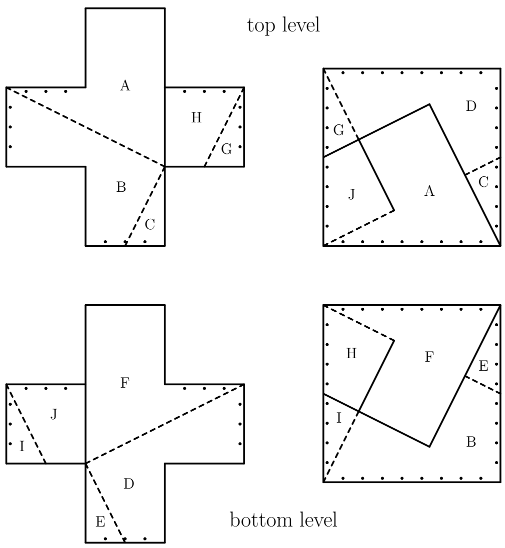
Two pieces side by side
(on the same level):



One piece on top of the other
(on different levels):

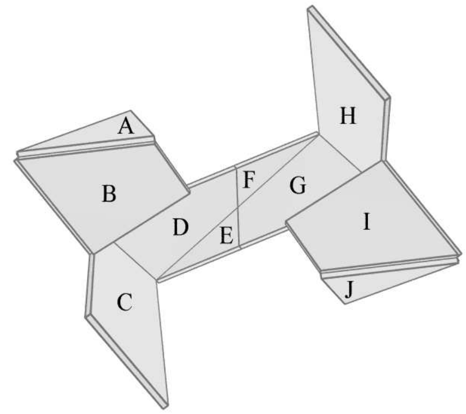


Greek cross to square [GNF, 2002]:



rotational symmetry

View of Greek cross to square:



three cycles of hinges

Cyclic Hingings

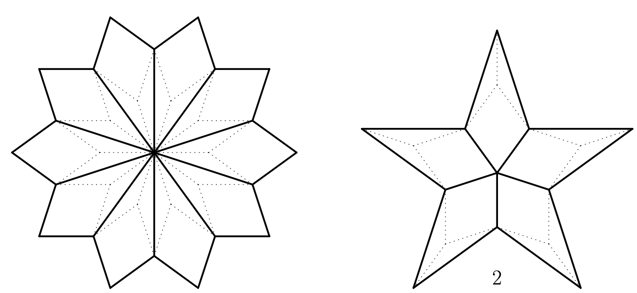
Vertex-cyclic hinging - when four or more pieces touch at a vertex and each piece is hinged with its predecessor and successor on the cycle.

- cap-cyclic hinging - when the angles that meet at the vertex sum to less than 360°
- flat-cyclic hinging - when the angles that meet at the vertex sum to exactly 360°
- saddle-cyclic hinging - when the angles that meet at the vertex sum to more than 360°

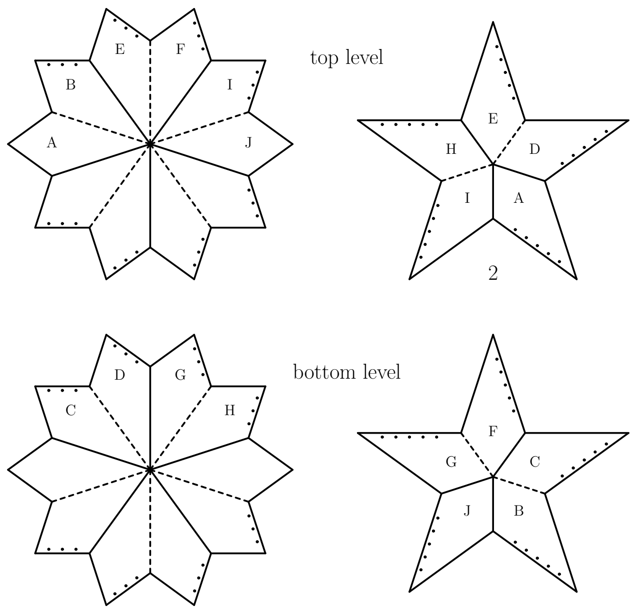
Tube-cyclic hinging

Leaf-cyclic hinging

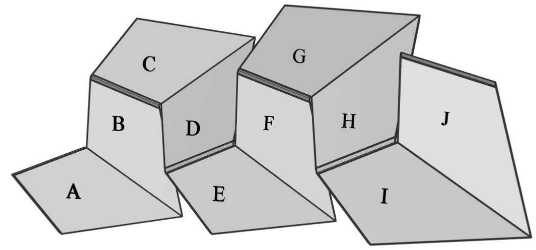
A {10/3} to two pentagrams (10 pieces) [Harry Lindgren, 1964]:



Fold-hinged dissection (20 pieces)
[GNF, 2001]:



View of fold-hinged pieces for the pentagram:



Note: Will NOT fold to a planar net!

Theorem. For any natural number n, there is an $(8n+4)$-piece piano-hinged dissection of two $\{(2n+1)/n\}$s to a $\{(4n+2)/(2n-1)\}$s.

(Real Conclusion)

There are many lovely examples

of piano-hinged dissections

— but that's another talk —

which will have to wait until next time!

Appreciation To:

Walt and Chris Hoppe -

Laser cutting wood and plexiglas models for the overhead projector

Wayne Daniel -

Crafting precise wooden models with real hinges